

FILTERED GEOMETRIC LATTICES AND LEFSCHETZ SECTION THEOREMS OVER THE TROPICAL SEMIRING

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ABSTRACT. The purpose of this paper is to establish analogues of the classical Lefschetz Section Theorem for smooth tropical varieties. More precisely, we prove tropical analogues of the section theorems of Lefschetz, Andreotti–Frankel, Bott–Milnor–Thom, Hamm–Lê and Kodaira–Spencer, and the vanishing theorems of Andreotti–Frankel and Akizuki–Kodaira–Nakano.

We start the paper by resolving a conjecture of Mikhalkin and Ziegler (2008) concerning the homotopy types of certain filtrations of geometric lattices, generalizing several known properties of full geometric lattices. This translates to a crucial index estimate for the stratified Morse data at critical points of the tropical variety; it can also by itself be interpreted as a Lefschetz-type theorem for matroids.

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Tropical geometry is a relatively young field in mathematics, based on early work of Bergman [Ber71] and Bieri–Groves [BG84]. It arises as algebraic geometry over the tropical max-plus semiring $\mathbb{T} = ([-\infty, \infty), \max, +)$, as well as a limit of classical complex algebraic geometry [GKZ94, Vir84]. Since tropical varieties are, in essence, polyhedral spaces, tropical geometry naturally connects the fields of algebraic and combinatorial geometry, and combinatorial tools have proven essential for the study of tropical varieties.

Since its origins, tropical geometry has been developed extensively [Gat06, RGST05, Spe05, SS09]. It has been applied to classical algebraic geometry [Gub07, Kat09], enumerative algebraic geometry [KT02, Mik05, Mik06, Shu05], mirror symmetry [Gro11, KS01], integrable systems [AMS12], and to several branches of applied mathematics, cf. [Gro95, NGVR12, Pin98]. Several classical results in algebraic geometry have natural analogues in tropical geometry, see e.g. [CDPR12, RGST05].

The purpose of this paper is to provide tropical analogues of one of the most central results in algebraic geometry, the Lefschetz Section Theorem (or Lefschetz Hyperplane Theorem). We attempt to give a comprehensive picture of the Lefschetz Section Theorem in tropical geometry, and give tropical analogues of many of the classical Lefschetz theorems and associated vanishing theorems. Along the way, we build on and generalize significant results in the topological theory of posets and matroids of Rota, Folkman, Quillen, Björner and others.

THE TROPICAL LEFSCHETZ SECTION THEOREMS.

The classical Lefschetz Section Theorem comes in many different guises. Intuitively, Lefschetz theorems relate the topology of a complex algebraic variety X to that of the intersection of X with a hyperplane H transversal to X (or, in abstract settings, to an ample divisor D of X). In its most classical form, it relates the homology groups of X and $X \cap H$ [Lef50, AF59] for smooth complex projective algebraic varieties.

Since Lefschetz pioneering work, many different variants of this important theorem were established. There are versions for affine varieties and projective varieties, for homology groups and homotopy groups, for Hodge groups and Picard groups, for constructible sheaves and several more; compare [GM88, Laz04, Voi02]. Via duality (Lefschetz duality, Morse duality, Serre duality etc.), Lefschetz theorems go hand-in-hand with vanishing theorems, such as the Andreotti–Frankel [AF59], Akizuki–Kodaira–Nakano [AN54] and Grothendieck–Artin [Laz04] Vanishing Theorems.

In this paper we establish analogues in tropical geometry of several of the classical Lefschetz theorems. More precisely, we shall provide tropical analogues of

- the Andreotti–Frankel Vanishing Theorem for affine varieties.
- the classical Lefschetz Section Theorem for homology groups of projective varieties, due to Lefschetz and Andreotti–Frankel [AF59, Lef50].
- the Bott–Milnor–Thom Lefschetz Section Theorem for homotopy groups and CW models of projective varieties [Bot59, Mil63].
- the Hamm–Lê Lefschetz Section Theorem for complements of affine varieties [Ham83, HL71].
- the Akizuki–Kodaira–Nakano Lefschetz Vanishing Theorem for Hodge groups [AN54, Voi02].
- the Kodaira–Spencer Lefschetz Section Theorem for Hodge groups [KS53].

Remarkably, our theorems apply even to situations that are not attainable as limits of the classical algebraic situation, and to matroids that are not realizable over any field. Compare this also to the Hard Lefschetz theorem for matroids (in a slightly more restricted setting) proven in [AHK].

Tropical Lefschetz Section Theorems for CW models, homotopy and homology. The first section theorem of this paper is an analogue of the Andreotti–Frankel Vanishing Theorem [AF59] for smooth tropical varieties.

Theorem 7.4. *Let $X \subset \mathbb{T}^d$ be a smooth affine n -dimensional tropical variety, and let H denote a tropical hypersurface in \mathbb{T}^d . Then X is, up to homotopy equivalence, obtained from $X \cap H$ by successively attaching n -dimensional cells.*

In particular, the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

Aside from the case of almost totally sedentary hyperplanes, we shall generally require that hypersurface complements in tropical space are pointed polyhedra. We also deduce a Lefschetz Section Theorem for projective tropical varieties (Theorem 7.8). Contrary to the original treatment of Andreotti–Frankel, this result does not follow immediately from Lefschetz duality and the affine theorem, but rather from a common generalization of the affine and projective cases (Lemma 7.7).

Tropical Lefschetz Section Theorems for complements of tropical varieties. Our reasoning extends to the complement of a tropical variety as well. This is analogous to the Hamm–Lê Lefschetz theorems [Ham83, HL71, Lê87] for complements of algebraic hypersurfaces.

Theorem 8.2. *Let X be a smooth n -dimensional tropical variety in \mathbb{T}^d , and let $C = C(X)$ be the complement of X in \mathbb{T}^d . Let furthermore H denote an almost totally sedentary hyperplane in \mathbb{T}^d transversal to X . Then C is, up to homotopy equivalence, obtained from $C \cap H$ by successively attaching $(d - n - 1)$ -dimensional cells.*

Here, an almost totally sedentary hyperplane is a hyperplane of the form $\mathbb{T}^{d-1} \times \{a\}$. The main tool to prove this result is the construction of an efficient Salvetti-type complex for complements of Bergman fans. In this connection we also characterize homotopically the “complement” of a matroid, see Corollary 3.7.

Tropical Lefschetz Section Theorems for tropical Hodge groups. Finally, we provide a Lefschetz theorem for tropical Hodge groups, or tropical (p, q) -homology, as defined by Itenberg–Katzarov–Mikhalkin–Zharkov [IKMZ]. This is nontrivial: While the classical Lefschetz Section Theorem for Hodge groups of smooth algebraic projective varieties (due to Kodaira–Spencer [KS53]) does follow from the Lefschetz Section Theorem for complex coefficients and the Hodge Decomposition, this approach does not apply here. Nevertheless, the Lefschetz Section Theorem holds true for tropical Hodge groups.

Theorem 10.4. *Let X be an n -dimensional smooth tropical variety in \mathbb{TP}^d , and let $H \subset \mathbb{TP}^d$ denote a hyperplane transversal to X . Then the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of (p, q) -homology as long as $p + q \leq n - 2$, and a surjection if $p + q = n - 1$.*

The theorem holds analogously for varieties in tropical affine space. For the proof we, instead of invoking a tropical Hodge Decomposition Theorem and Theorems 7.4 and 7.8, establish a tropical analogue of the Akizuki–Kodaira–Nakano Vanishing Theorem.

Theorem 10.5. *Let X denote any n -dimensional smooth tropical variety in \mathbb{TP}^d , and let P denote any pointed d -polyhedron in \mathbb{TP}^d . Then*

$$H_q(X \cap P, X \cap \partial_{\text{in}} P; \mathcal{F}_p X) = 0 \quad \text{for all } p + q < n.$$

The proof of the tropical Kodaira–Spencer Theorem can then be finished in a manner similar to the classical proof via the long exact sequence of Hodge groups, compare also [AN54, Voi02]. We also provide a counterexample to the integral version of the above theorem.

FILTERED GEOMETRIC LATTICES.

For the proofs of the tropical Lefschetz theorems, we shall critically use stratified Morse theory ([GM88], see also our brief introduction to stratified Morse theory in Section 4). A crucial ingredient of the Morse-theoretic approach to classical Lefschetz Theorems are estimates on Morse indices at critical points, which follow easily from general considerations on Hessians of homogeneous complex polynomials, cf. [AF59, Laz04, Mil63].

In our setting, we analogously estimate the topological changes in the sublevel sets with respect to some smooth Morse function, interpreting the tropical variety as a Whitney stratified space. This requires us to verify a conjecture of Mikhalkin and Ziegler [MZ08] concerning topological properties of geometric lattices. The terminology and notation used here is explained in Section 2.

Theorem 2.1. *Let \mathcal{L} denote the lattice of flats of a matroid on groundset $[n]$ and of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t denote any real number with $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t}$ is homotopy Cohen–Macaulay of dimension $r - 2$. In particular, it is $(r - 3)$ -connected.*

Geometrically, this result implies a topological characterization of half-links of a smooth tropical variety at critical points. For the proof, we rely on lexicographic shellability and a generalization of Quillen’s fiber lemma.

Moreover, Theorem 2.1 generalizes earlier characterizations of the topological type of the full geometric lattice \mathcal{L} , the homology version of which goes back to work of Folkman [Fol66], inspired by work of Rota [Rot64] on the Möbius function of geometric lattices. A stronger version concerning shellability, and therefore also homotopy equivalence, was later proved by Björner [Bjö80].

Other than for full geometric lattices, Theorem 2.1 was previously known for Boolean lattices [Bjö13] (equivalently: free matroids), and for lattices of rank 3 [PZ08], and also for the case when the weight ω has only one negative entry (this is implied by a result of Wachs and Walker [WW86]).

PLAN FOR THE PAPER.

In Section 2 we prove our main theorem on the homotopy Cohen–Macaulayness of filtered geometric lattices, using methods from poset topology. Also, in Remark 3.2 we sketch an alternative proof based on the combinatorial Morse Theory of Forman [For98].

In Sections 7, 8 and 10 we apply our results to deriving Lefschetz Theorems for tropical varieties. In each section, we first review a classical Lefschetz theorem, then proceed to give a tropical analogue.

In the remaining sections we review and extend required background information, on combinatorial, cellular and poset topology, geometry and combinatorics of polyhedral complexes, tropical geometry, tropical Hodge theory, and stratified Morse theory. While the main purpose for this material is to provide a carefully laid foundation for the proofs of the main results, it is also meant to provide a coherent presentation of the material.

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Part A. Filtered Geometric lattices

1. SOME BASIC COMBINATORIAL TOPOLOGY.

We recall some basic facts from algebraic topology and the topology of posets. The reader is referred to [Mun84], [Hat02], [Bjö95] and [Whi78] for more details. All topological spaces have the homotopy type of simplicial complexes and, in particular, always have a CW decomposition.

1.1. Polyhedral spaces and complexes. A *(closed) polyhedral complex* in \mathbb{R}^d is a finite collection of polyhedra in \mathbb{R}^d such that the intersection of any two polyhedra is a face of both, and that is closed under passing to faces of the polyhedra in the collection. The elements of a polyhedral complex are called *faces*, and the inclusion-wise maximal faces are the *facets* of the polyhedral complex. A polyhedral complex is *bounded* if and only if all polyhedra are bounded, i.e., if they are polytopes. Finally, a *polyhedral fan* is a polyhedral complex all whose faces are polyhedra pointed at $\mathbf{0}$. $\text{pos } X$ shall denote the positive span of a subset X of \mathbb{R}^d , and $\text{lin } X$ resp. $\text{aff } X$ shall denote its linear and affine span, respectively.

The *underlying (polyhedral) space* $|X|$ of a polyhedral complex X is the union of its faces. With abuse of notation, we often speak of the polyhedral complex when we actually mean its underlying space. We define the *combinatorial restriction* $X|_M$ of a polyhedral complex X to a set M to be the inclusion-wise maximal subcomplex D of X such that $D \subset M$. Finally, the *deletion* $X - D$ of a subcomplex D from X is the subcomplex of X given by $X|_{X \setminus D^\circ}$.

If X and Y are two polyhedral complexes with the same underlying space, then Y is called a *refinement* or *subdivision* of X if every face y of Y is contained in some face x of X . Similarly, for polyhedral complexes X, Y we define the *common refinement* $X \cdot Y$ as the polyhedral complex $\{x \cap y : x \in X, y \in Y\}$, so that in particular $|X \cdot Y| = |X| \cap |Y|$.

1.2. Acyclicity, Connectivity and Cohen-Macaulayness. A topological space X is said to be *k-connected* if one of the following equivalent conditions holds:

- $\pi_i(X) = 0$ for all $i \leq k$, i.e., every map of the sphere S^i , $i \leq k$, into Δ is null-homotopic,
- X is homotopy equivalent to a CW complex that, except for the basepoint, has no cells of dimension $\leq k$.

Similarly, a pair of topological spaces (X, Y) is *k-connected* if $\pi_i(X, Y) = 0$ for all $i \leq k$.

A space X is *k-acyclic* if $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \leq k$, and a pair of spaces (X, Y) is *k-acyclic* if $\tilde{H}_i(X, Y; \mathbb{Z}) = 0$ for all $i \leq k$. Every *k-connected* space is *k-acyclic*, by elementary cellular homology. We will repeatedly make use of the fact that by the Theorems of Whitehead and Hurewicz (see for instance [Hat02, Section 4]), a *k-acyclic* space (or pair of spaces), $k \geq 1$, is *k-connected* if and only if it is 1-connected.

A pure simplicial complex Δ of dimension $d - 1$ is *homotopy Cohen-Macaulay* if any of the following equivalent conditions holds

- for all faces σ in Δ , the link $\text{lk}_\sigma \Delta$ is $(d - \dim \sigma - 3)$ -connected.
- for all faces σ in Δ , the link $\text{lk}_\sigma \Delta$ is homotopy equivalent to a wedge of $(d - \dim \sigma - 2)$ -dimensional spheres.

Here the empty set is considered to be a (-1) -dimensional face, and $\text{lk}_\emptyset \Delta = \Delta$.

A pure $(d - 1)$ -dimensional simplicial complex Δ is *Cohen-Macaulay over \mathbb{Z}* if $\tilde{H}_i(\text{lk}_\sigma \Delta; \mathbb{Z}) = 0$ for all faces $\sigma \in \Delta$ and all $i < \dim \text{lk}_\sigma \Delta = d - \dim \sigma - 2$. Being Cohen-Macaulay over R is similarly defined for other coefficient rings R . See [Sta96] for some of the algebraic ramifications of this concept.

1.3. Elementary cellular topology. We use $A * B$ to denote the join of two topological spaces (CW complexes, simplicial complexes) A, B , and $\mathcal{C}X \stackrel{\text{def}}{=} \{\text{point}\} * X$ to denote the (abstract) cone over a topological space X .

If X is any topological space, and $Y \subset X$ is any subspace, then we say that X is obtained from Y by attaching an i -cell if X can, up to homotopy equivalence, be decomposed as the union

$$Y \cup e / \alpha(x) \sim x$$

where e is an i -cell and α is a continuous inclusion $\partial e \rightarrow Y$.

We now recall three well-known results in combinatorial topology.

Lemma 1.1. *Let Δ and $\Gamma \subset \Delta$ denote a pair of simplicial complexes. Then $\Delta \setminus \Gamma$ deformation retracts to $\Delta - \Gamma$.* \square

Lemma 1.2. *Let Δ, Γ denote two topological spaces that are k -connected and ℓ -connected, respectively. Then $\Delta * \Gamma$ is $(k + \ell + 2)$ -connected.*

Proof. Let us consider the spaces $\Delta' = \Delta * \Gamma \setminus \Gamma$ and $\Gamma' = \Delta * \Gamma \setminus \Delta$. Then $\Delta' \simeq \Delta$, $\Gamma' \simeq \Gamma$ and $\Delta' \cap \Gamma' \simeq \Delta \times \Gamma$. By considering the Mayer–Vietoris sequence for Δ' and Γ' , together with the Künneth formula, we see that $\Delta * \Gamma$ is $(k + \ell + 2)$ -acyclic. The claim follows from the Whitehead and Hurewicz Theorems. \square

Lemma 1.3. *Let Δ denote a polytopal complex, and let σ be any ℓ -cell of Δ . If $\text{lk}_\sigma \Delta$ is k -connected, then Δ is, up to homotopy equivalence, obtained from $\Delta - \sigma$ by successively attaching cells of dimension $\geq k + \ell + 2$.*

Proof. By a stellar subdivision at σ and Lemma 1.2, it suffices to address the case $\ell = 0$, i.e., the case when $\sigma = v$ is a vertex. We may furthermore assume that $k \geq 0$, since the claim is trivial otherwise.

Let K denote a CW complex homotopy equivalent to $\text{lk}_v \Delta$ and constructed so that it has no nontrivial cells of dimension $\leq k$ (i.e. no cells apart from the basepoint). Let $f : K \rightarrow \text{lk}_v \Delta$ denote a continuous mapping realizing the homotopy equivalence $K \simeq \text{lk}_v \Delta$, and let

$$M_f = K \times [0, 1] \cup \text{lk}_v \Delta / (x, 0) \sim f(x)$$

denote its mapping cylinder. Then Δ is homotopy equivalent to

$$((\Delta - v) \cup M_f) \cup \mathcal{C}(K) / x \in \partial(\mathcal{C}(K)) \sim (x, 1)$$

Now, if c is any nontrivial cell of $\partial(\mathcal{C}(K))$, then $\mathcal{C}(c)$ is a disk in $\mathcal{C}(K)$ of dimension $\geq k + 2$ (since c is a cell of dimension $\geq k + 1$). Since all nontrivial cells are of this form, the claim follows. \square

1.4. Topology of posets. Posets \mathcal{P} are interpreted topologically via their *order complex* $\Delta(\mathcal{P})$, whose faces are the totally ordered subsets (chains) of \mathcal{P} . Here $\Delta(\cdot)$ is usually suppressed from the notation. For instance, for a $(d - 1)$ -dimensional homotopy Cohen–Macaulay poset \mathcal{P} as above, we have $\mathcal{P} \simeq \bigvee S^{d-1}$. As a general reference for poset topology, see [Bjö95].

A well-known consequence of Lemma 1.2 (see e.g. [Qui78, Bjö95]) is that a poset is Cohen–Macaulay (resp. homotopy CM) if and only if its intervals of length k are $(k - 1)$ -acyclic (resp. $(k - 1)$ -connected) for all k .

For a poset \mathcal{P} and two comparable elements $a, b \in \mathcal{P}$, we have the *interval* $\mathcal{P}_{[a,b]} \stackrel{\text{def}}{=} \{y \in \mathcal{P} : a \leq y \leq b\}$ (and similarly for open and half-open intervals). We also have the *lower* (resp. *upper*) *ideal* $\mathcal{P}_{\leq x} = \{y \in \mathcal{P} : y \leq x\}$ (resp. $\mathcal{P}_{\geq x} = \{y \in \mathcal{P} : y \geq x\}$) of an element $x \in \mathcal{P}$. Lower ideals are also called *order filters* in the literature.

An order-preserving map $f : \mathcal{P} \rightarrow \mathcal{P}$ is called a *closure operator* if $x \leq f(x) = f^2(x)$ for all $x \in \mathcal{P}$. One can deduce from Lemma 1.4 below that such a map induces homotopy equivalence of \mathcal{P} and its image $f(\mathcal{P})$. But more is true: a closure operator is a strong deformation retract. See also [Bjö95, p. 1852] and the proof of Lemma 1.4 below.

A concrete example of a closure operator that plays a role in this paper is the closure map of a matroid, sending an arbitrary set of points to the smallest closed set containing it. A homotopy inverse is the identity map, sending a closed set to itself.

A central tool for our line of reasoning is Quillen's "Theorem A", which we now give in a version that is somewhat more general than those available in the literature, cf. [Qui78, Bjö03, BWW05].

Lemma 1.4. *Let \mathcal{P} , \mathcal{Q} be two posets, and $\varphi : \mathcal{P} \rightarrow \mathcal{Q}$ an order-preserving map. Assume that for every $x \in \mathcal{Q}$, the fiber $\varphi^{-1}(\mathcal{Q}_{\leq x})$ is m_x -connected and $\mathcal{Q}_{> x}$ is ℓ_x -connected, and let*

$$k \stackrel{\text{def}}{=} \min_{x \in \mathcal{Q}} (m_x + \ell_x) + 2.$$

Then \mathcal{Q} is, up to homotopy equivalence, obtained from \mathcal{P} by attaching cells of dimension $\geq k + 2$.

Consequently,

- (1) *φ induces isomorphisms of homotopy groups up to dimension k , and a surjection in dimension $k + 1$.*
- (2) *\mathcal{P} is k -connected if and only if \mathcal{Q} is k -connected.*

Proof. Let us consider the poset M_φ whose ground set is the disjoint union of the elements of \mathcal{P} and \mathcal{Q} , and where we define

- for $q, q' \in \mathcal{Q} \subset M_\varphi$, then $q \leq q'$ in M_φ if and only if $q \leq q'$ in \mathcal{Q} ,
- for $p, p' \in \mathcal{P} \subset M_\varphi$, then $p \leq p'$ in M_φ if and only if $p \leq p'$ in \mathcal{P} , and
- for $p \in \mathcal{P} \subset M_\varphi$ and $q \in \mathcal{Q} \subset M_\varphi$, then $p \leq q$ in M_φ if and only if $\varphi(p) \leq q$ in \mathcal{Q} .

The poset M_φ triangulates the mapping cylinder of φ , and therefore strongly deformation retracts to \mathcal{Q} . Moreover, if $\tilde{\varphi}$ denotes the inclusion map $\mathcal{P} \hookrightarrow M_\varphi$, then for every $x \in \mathcal{Q} \subset M_\varphi$, we have the isomorphisms

$$\varphi^{-1}(\mathcal{Q}_{\leq x}) \cong \tilde{\varphi}^{-1}(\mathcal{Q}_{\leq x}) \quad \text{and} \quad \mathcal{Q}_{> x} \cong (M_\varphi)_{> x}.$$

The key observation now is that we can obtain \mathcal{P} from M_φ by removing the elements of $\mathcal{Q} \subset M_\varphi$ one by one, until only \mathcal{P} is left. We do so in an increasing fashion, removing the elements from bottom to top.

To make this precise, let \mathcal{I} denote any poset $\mathcal{P} \subsetneq \mathcal{I} \subset M_\varphi$, and let μ denote a minimal element of $\mathcal{I} \setminus \mathcal{P}$, such that $\mathcal{I}_{\geq \mu} = (M_\varphi)_{\geq \mu} = \mathcal{Q}_{\geq \mu}$, i.e., no element greater than μ has yet been deleted from $(M_\varphi)_{\geq \mu}$.

Now, $\text{lk}_\mu \mathcal{I} \cong \mathcal{Q}_{> \mu} * \varphi^{-1}(\mathcal{Q}_{\leq \mu})$, is k -connected by assumption and Lemma 1.2. Hence \mathcal{I} is obtained from $\mathcal{I} \setminus \{\mu\}$ by successively attaching cells of dimension $\geq k + 2$, by Lemma 1.3. By extension, $M_\varphi \simeq \mathcal{Q}$ is obtained from \mathcal{P} by successively attaching cells of dimension $\geq k + 2$. The first claim follows, and this implies the other two. \square

2. FILTERED GEOMETRIC LATTICES

This section is devoted to proving the conjecture of Mikhalkin and Ziegler [MZ08] about the lattice of flats of a weighted matroid. We assume familiarity with the basic properties of matroids and geometric lattices, see e.g. [Oxl11]. For the homological aspects, see [Bjö92].

Let M denote a matroid on the ground set $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$. As a general convention, we shall assume that all matroids are loopless. A *weight* $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ on M is any vector in $\mathbb{R}^{[n]}$. If σ is any subset of $[n]$, and $\mathbf{1}_\sigma$ is its characteristic vector, then we set

$$\omega \cdot \sigma \stackrel{\text{def}}{=} \omega \cdot \mathbf{1}_\sigma = \sum_{e \in \sigma} \omega_e.$$

A weight is *generic* if $\omega \cdot \sigma \neq 0$ for all proper subsets $\emptyset \subsetneq \sigma \subsetneq [n]$. If $\mathcal{L} = \mathcal{L}[M] \stackrel{\text{def}}{=} \widehat{\mathcal{L}} \setminus \{\widehat{0}, \widehat{1}\}$ is the proper part of the lattice of flats $\widehat{\mathcal{L}}$ of M , and t is any real number, then we use $\mathcal{L}^{>t}$ to denote the subset of \mathcal{L} consisting of elements $\sigma \in \mathcal{L}$ with $\omega \cdot \sigma > t$. We will refer to the posets (partially ordered sets) of the form $\mathcal{L}^{>t}$ as *filtered geometric lattices*. Note that these posets are not lattices in general, let alone geometric lattices. Nevertheless, we can state and subsequently establish the main result of this section.

Theorem 2.1. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t denote any real number with $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t}$ is homotopy Cohen–Macaulay of dimension $r - 2$, and in particular $(r - 3)$ -connected.*

The proof of this result is articulated in a few steps. We start from homotopy information available for free matroids, and from this we deduce information concerning $\mathcal{L}^{>t}$, using our generalization of Quillen’s “Theorem A” (Lemma 1.4). As an immediate corollary, we obtain what can reasonably be called a “Lefschetz Section Theorem for matroids”:

Corollary 2.2. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t, t' be any pair of real numbers with $t' < t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{L}^{>t'}$ is obtained from $\mathcal{L}^{>t}$ by attaching cells of dimension $r - 2$. In particular, $(\mathcal{L}^{>t'}, \mathcal{L}^{>t})$ is homotopy Cohen–Macaulay of dimension $r - 2$.*

Proof. This follows directly from Theorem 2.1, the long exact sequence of relative homotopy groups, and Lemma 1.3. \square

It is instructive to consider the case when $\omega \cdot [n] = 0$ and $t = 0$. Then the geometric lattice splits into two parts $\mathcal{L}^{>0}$ and $\mathcal{L}^{<0}$, which are both homotopy Cohen–Macaulay and of the same dimension. Moreover, both can be thought of as the complement of a geometric hyperplane in \mathcal{L} via the Bergman fan, compare Section 6.

For full geometric lattices it is known from the work of Rota [Rot64] on the Möbius function that the number of $(r - 2)$ -spheres in the wedge is strictly positive. This is not true for filtered geometric lattices. For example, if there is exactly one positive weight $\omega_i > 0$ then $\mathcal{L}^{>0}$ is contractible. However, the following relative information is immediate in the general case:

Corollary 2.3. *If $t' < t \leq \min\{0, \omega \cdot [n]\}$, then*

$$\dim(H_{r-2}(\mathcal{L}^{>t})) \leq \dim(H_{r-2}(\mathcal{L}^{>t'})).$$

2.1. Preliminaries to the proof. Let us first observe a general heredity property of filtered geometric lattices that we will use repeatedly for purposes of induction.

Lemma 2.4. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 2.1. Let $(\mathcal{L}^{>t; \omega})_{(\sigma, \tau)}$ be any open interval in $\mathcal{L}^{>t}$. Then, for $t' = t - \omega \cdot \sigma$ and $\omega' = \omega|_{(\tau - \sigma)}$, we have*

$$(\mathcal{L}^{>t; \omega})_{(\sigma, \tau)} = (\mathcal{L}_{(\sigma, \tau)})^{>t'; \omega'}.$$

Proof. Consider first the case when $\sigma = \emptyset$. Then $\mathcal{L}_{(\sigma, \tau)} = \mathcal{L}_{<\tau}$ is the lattice of flats of the matroid M' of rank $\text{rk}(\tau)$ obtained as the restriction of M to τ . Therefore, $\mathcal{L}_{<\tau}^{>t} \cong \mathcal{L}[M']^{>t}$, where M' is endowed with the weight given by the restriction $\omega|_{\tau}$ of ω to the set τ .

Next, suppose that $\tau = [n]$. Then $\mathcal{L}_{(\sigma, \tau)} = \mathcal{L}_{>\sigma}$ is the lattice of flats of the rank $(r - \text{rk}(\sigma))$ matroid M'' obtained as the contraction of σ in M . Moreover, if M'' is endowed with weight $\omega|_{[n] \setminus \sigma}$, we have

$$\mathcal{L}_{>\sigma}^{>t} \cong \mathcal{L}[M'']^{>(t - \omega \cdot \sigma)}.$$

Since $\mathcal{L}_{(\sigma, \tau)} = (\mathcal{L}_{<\tau})_{>\sigma}$, the general result is obtained from these two special cases. \square

The fact that all maximal chains in $\mathcal{L}^{>t}$ have equal length $r - 2$ is a direct consequence.

Lemma 2.5. $\mathcal{L}^{>t}$ is pure and $(r - 2)$ -dimensional.

Proof. For rank $r = 2$ the statement boils down to saying that $\mathcal{L}^{>t}$ is nonempty. Suppose that this were not the case. Then $\omega_i \leq t$ for all i , implying that $t \leq \omega \cdot [n] \leq tn$, which is impossible if $t < 0$. The case when $t = 0 = \omega \cdot [n]$ is clear.

A proof by induction on rank now follows easily from Lemma 2.4 by considering intervals $\mathcal{L}_{>\sigma}^{>t}$ where σ is an atom. \square

2.2. Free matroids. We begin with the following strengthening of Theorem 2.1 for the special case of free matroids, that is, matroids where all sets are independent.

We reserve the notation $\mathcal{B} = \mathcal{B}[n]$ for the proper part of the lattice of flats of the free matroid on n elements. It coincides with the proper part of the Boolean lattice $\widehat{\mathcal{B}} = 2^{[n]}$ of subsets of $[n] = \{1, \dots, n\}$, that is, $\mathcal{B} = 2^{[n]} \setminus \{\emptyset, [n]\}$.

Theorem 2.6. Let ω denote any generic weight on $[n]$, and suppose that $t \leq \min\{0, \omega \cdot [n]\}$. Then $\mathcal{B}^{>t}$ is shellable and $(n - 2)$ -dimensional. In particular, it is homotopy Cohen–Macaulay.

Proof. We use the method of lexicographic shellability [Bjö80, Bjö13]. We may assume that $\omega_i \neq \omega_j$ for $i \neq j$. This can always be achieved by a small perturbation of the weight vector ω that does not change $\mathcal{B}^{>t}$.

To each covering edge (σ, τ) of $\widehat{\mathcal{B}}$ we assign the real number $\lambda(\sigma, \tau) = \omega \cdot (\tau \setminus \sigma) = \omega \cdot \tau - \omega \cdot \sigma$. This edge labeling induces a labeling of the maximal chains of $\widehat{\mathcal{B}}^{>t}$. We know from Lemma 2.5 that these chains are all of cardinality $n + 1$ (including the top and bottom elements \emptyset and $[n]$). The label $\lambda(m)$ of a maximal chain m is simply the induced permutation of the coordinates of the weight vector ω .

There is a unique maximal chain \overline{m} in $\widehat{\mathcal{B}}$ with the property that the labels form a decreasing sequence. After relabeling this is

$$\lambda(\overline{m}) = (\omega_1 > \omega_2 > \dots > \omega_n).$$

We have that

$$\emptyset \in \widehat{\mathcal{B}}^{>t} \iff t < 0 \quad \text{and} \quad [n] \in \widehat{\mathcal{B}}^{>t} \iff t < \omega \cdot [n],$$

so the hypothesis $t \leq \min\{0, \omega \cdot [n]\}$ implies that both endpoints of the chain \overline{m} belong to $\widehat{\mathcal{B}}^{>t}$. From this follows that the entire chain \overline{m} is in $\widehat{\mathcal{B}}^{>t}$, as is easy to see. Also, this chain is lexicographically first among the maximal chains in $\widehat{\mathcal{B}}$, and so also in $\widehat{\mathcal{B}}^{>t}$.

Similar reasoning can be performed locally at each interval (μ, ν) to prove the existence of a unique decreasingly labeled maximal chain in (μ, ν) which lexicographically precedes all the other maximal chains in that interval. This completes the verification of the conditions for lexicographic shellability. \square

Remark 2.7. The conclusion of the theorem can be sharpened to state that $\mathcal{B}^{>t}$ is homeomorphic to a ball or a sphere. Some aspects of this additional information are discussed in [Bjö13], it will not be needed here.

2.3. Connectivity and Cohen–Macaulayness. We continue the proof of Theorem 2.1 by establishing the degree of connectivity for $\mathcal{L}^{>t}$.

Theorem 2.8. Let $(\mathcal{L}, r, n, \omega, t)$ be as in the statement of Theorem 2.1. Then $\mathcal{L}^{>t}$ is $(r - 3)$ -connected.

Proof. We prove the theorem by induction on the cardinality $n = |M|$, the case $n = 1$ being trivial.

In hope of being able to apply a Quillen-type fiber argument, let us consider the inclusion map $\varphi : \mathcal{L}^{>t} \hookrightarrow \mathcal{B}^{>t}$ and analyze the fibers $\varphi^{-1}(\mathcal{B}_{\geq x}^{>t})$ and the lower ideals $\mathcal{B}_{< x}^{>t}$, for all $x \in \mathcal{B}^{>t}$.

We have that $t < \omega \cdot x$, since $x \in \mathcal{B}^{>t}$, and $t \leq \min\{0, \omega \cdot [n]\}$. Hence, $t \leq \min\{0, \omega \cdot x\}$, and it follows from Lemma 2.4 and Theorem 2.6 that the posets $\mathcal{B}_{\leq x}^{>t} \cong \mathcal{B}[x]^{>t}$ are $(|x| - 3)$ -connected.

It remains to consider the fibers $\varphi^{-1}(\mathcal{B}_{\geq x}^{>t})$. Let $\kappa : \mathcal{B} \rightarrow \mathcal{L}$ denote the matroid closure map $S \mapsto \bigvee_{e \in S} e$, and let x be any element in $\mathcal{B}^{>t}$. Then,

$$\varphi^{-1}(\mathcal{B}_{\geq x}^{>t}) = \mathcal{L}_{\geq x}^{>t} = \mathcal{L}_{\geq \kappa(x)}^{>t}.$$

If $\kappa(x) \in \mathcal{L}^{>t}$, the fiber is a cone, and hence contractible.

If $\kappa(x) \notin \mathcal{L}^{>t}$, then by the induction assumption and Lemma 2.4, $\mathcal{L}_{\geq \kappa(x)}^{>t}$ is $(\dim \mathcal{L}_{\geq \kappa(x)}^{>t} - 1)$ -connected.

We have shown that

the fiber $\varphi^{-1}(\mathcal{B}_{\geq x}^{>t})$ is $(\dim \mathcal{L}_{\geq \kappa(x)}^{>t} - 1)$ -connected and the ideal $\mathcal{B}_{< x}^{>t}$ is $(|x| - 3)$ -connected,

for all $x \in \mathcal{B}^{>t}$. Hence, by Fiber Lemma 1.4, the inclusion map φ yields an isomorphism of homotopy groups up to dimension k , and a surjection in dimension $k + 1$, where

$$k \stackrel{\text{def}}{=} \min_{\substack{x \in \mathcal{B}^{>t} \\ \kappa(x) \notin \mathcal{L}^{>t}}} (\dim(\mathcal{L}_{\geq \kappa(x)}^{>t}) + |x|) - 2$$

Now, for $x \in \mathcal{B}^{>t}$ with $\kappa(x) \notin \mathcal{L}^{>t}$, we have

$$\begin{aligned} \dim \mathcal{L}_{\geq \kappa(x)}^{>t} + |x| - 2 &\geq \dim \mathcal{L}_{\geq \kappa(x)}^{>t} + \dim(\mathcal{L}_{\leq \kappa(x)}^{>t}) - 1 \\ &= \dim \mathcal{L}^{>t} - 1 \\ &= r - 3 \end{aligned}$$

Hence, $k \geq r - 3$, and since $\mathcal{B}^{>t}$ is $(r - 3)$ -connected, so is $\mathcal{L}^{>t}$. □

We can now finish and prove the homotopy Cohen–Macaulay property, which demands that we show the purity of $\mathcal{L}^{>t}$ and that each interval is connected up to its dimension minus one.

Proof of Theorem 2.1. Let $(\mathcal{L}^{>t})_{(\sigma, \tau)}$ be an open interval. We know from Lemma 2.5 that its order complex has dimension $\text{rk}(\tau) - \text{rk}(\sigma) - 2$ and from Lemma 2.4 and Theorem 2.8 that it is $(\text{rk}(\tau) - \text{rk}(\sigma) - 3)$ -connected. □

3. FILTERED GEOMETRIC LATTICES: REMARKS, EXAMPLES AND OPEN PROBLEMS.

3.1. Shellability. It remains to be seen whether our understanding of the combinatorial structure of filtered geometric lattices can be improved further. The conjecture of Mikhalkin and Ziegler [MZ08] actually predicts the stronger property of shellability.

Open Problem 3.1. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 2.1. Is it true that $\mathcal{L}^{>t}$ is shellable?*

A positive answer would generalize earlier work of Björner [Bjö80] showing that every full geometric lattice is shellable, and by Wachs and Walker [WW86] dealing with the case when the weight ω has exactly one negative entry.

3.2. Combinatorial Morse Theory. For an alternative approach to the conjecture of Mikhalkin and Ziegler one can use the combinatorial Morse theory of Forman [For98]. Intuitively speaking, combinatorial Morse theory is an incremental way to decompose a simplicial complex step-by-step that enriches Whitehead's notion of cell collapses [Whi78] by the notion of *critical cells*, which behave analogously to critical points in classical Morse Theory. The result is:

Theorem 3.2. *Let $(\mathcal{L}, r, n, \omega)$ be as in Theorem 2.1, with $\omega \cdot [n] = 0$. Then, there is a collection C of critical $(r - 2)$ -cells such that $\mathcal{L}^{>0} - C$ simplicially collapses to a point. In particular, $\mathcal{L}^{>0}$ is $(r - 3)$ -connected.*

In comparison with Theorem 2.1, this result requires a stronger assumption (the total weight of ω is 0). But it also has a stronger conclusion, since it describes the combinatorial structure of $\mathcal{L}^{>0}$ and not only its topological type. For the proof, one uses Alexander duality of combinatorial Morse functions as introduced in [Adi12]. This can be exploited to prove Theorem 3.2 and the analogous theorem for the complement of $\mathcal{L}^{>0}$ in \mathcal{B} by a common induction.

3.3. General filtered geometric lattices. Towards a complete understanding of filtered geometric lattices, it remains to consider the case when the filtration parameter does not satisfy $t \leq \min\{0, \omega \cdot [n]\}$.

Open Problem 3.3. *Characterize the topology of $\mathcal{L}^{>t}$ for general t .*

It would seem natural to expect that $\mathcal{L}^{>t}$ is always *sequentially Cohen–Macaulay*, a notion introduced by Stanley and Björner–Wachs to generalize Cohen–Macaulayness to nonpure complexes, cf. [BWW09, Sta96]. This, however, is not the case.

Example 3.4. Let us consider the matroid M on ground set $[7]$, endowed with lattice of flats

$$\mathcal{L} \stackrel{\text{def}}{=} \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{1, 2\}, \{6, 7\}, \{1, 3, 6\}, \{1, 4, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}\},$$

see Figure 3.1. Let us furthermore consider the weight $\omega = (1, 1, -3, -3, -3, 1, 1)$. Then

$$\mathcal{L}^{>0} = \{\{1\}, \{2\}, \{6\}, \{7\}, \{1, 2\}, \{6, 7\}\},$$

which consists of two disconnected 1-dimensional complexes. Hence, $\mathcal{L}^{>0}$ is not sequentially connected, and in particular not sequentially Cohen–Macaulay.

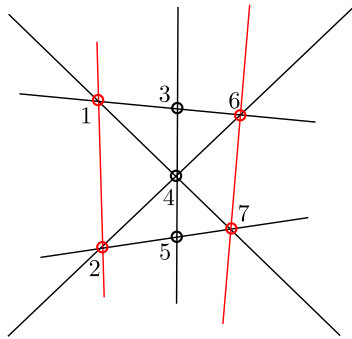


Figure 3.1. The matroid M , its proper flats and the filtered geometric lattice $\mathcal{L}^{>0}$ in red.

3.4. The complement of a filtered geometric lattice. We have established that \mathcal{B} is obtained from $\mathcal{B}^{>t}$ (Theorem 2.6), and that $\mathcal{B}^{>t}$ is obtained from $\mathcal{L}^{>t}$ (Theorem 2.8), by successively attaching cells of dimension $\geq r - 2$. One can reverse the reasoning to prove the following theorem:

Theorem 3.5. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 2.1. Then $\mathcal{B} - \mathcal{L}$ is obtained from $\mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}$ by attaching cells of dimension $\leq n - r - 1$.*

The proof is entirely analogous to that of Theorem 2.8, and will be left out here. We notice, however, several facts:

(1) By Lemma 1.1 and Alexander/Poincaré-Lefschetz duality, we have an isomorphism

$$H_i(\mathcal{B} - \mathcal{L}, \mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}) \cong H^{n-i-3}(\mathcal{L}^{>t}).$$

Theorem 3.5 therefore provides an alternative proof for the homology version of Theorem 2.8 (and vice versa).

(2) Furthermore, if $n - r \neq 2$, Theorems 2.8 and 3.5 are equivalent by well-known general position arguments together with the aforementioned duality.

(3) It is possible to give a common proof of Theorem 2.8 and Theorem 3.5, using combinatorial Morse theory and Alexander duality of combinatorial Morse functions, cf. Section 3.2.

(4) The pair $(\mathcal{B} - \mathcal{L}, \mathcal{B}^{\leq t} - \mathcal{L}^{\leq t})$, being the complement of an $(r - 2)$ -dimensional complex \mathcal{L} in the $(n - 3)$ -connected, $(n - 2)$ -dimensional pair $(\mathcal{B}, \mathcal{B}^{\leq t})$, is $(n - r - 2)$ -connected by classical general position arguments. Together with the information that the pair is of dimension $\leq n - r - 1$, we immediately obtain the following more precise version of Theorem 3.5:

Corollary 3.6. *Let $(\mathcal{L}, r, n, \omega, t)$ be as in Theorem 2.1. Then $\mathcal{B} - \mathcal{L}$ is obtained from $\mathcal{B}^{\leq t} - \mathcal{L}^{\leq t}$ by attaching $(n - r - 1)$ -dimensional cells.*

In particular, we can extend the results on the homotopy type of geometric lattices to their complements.

Corollary 3.7. *Let \mathcal{L} denote the lattice of flats of a matroid of rank $r \geq 2$, and let \mathcal{B} denote the proper part of the Boolean lattice on the ground set of M . Then $\mathcal{B} - \mathcal{L}$ is homotopy equivalent to a wedge of spheres of dimension $|M| - r - 1$.*

3.5. An efficient model for complements of geometric lattices. While we now understand, from a homotopical point of view, the complement of a geometric lattice in the Boolean lattice on the same support, it might be desirable to have a more explicit model available. For this purpose, we can use an idea similar to Salvetti [Sal87] and Björner–Ziegler [BZ92], who described models for the complement of subspace arrangements. Throughout this remark, we use $(M, \mathcal{L}, \mathcal{B}, r, n)$ as in the previous sections.

A naive model for the complement $\mathcal{B} \setminus \mathcal{L}$ of \mathcal{L} in \mathcal{B} is clearly given by the complex $\mathcal{B} - \mathcal{L}$, as shown by Lemma 1.1. However, the complex $\mathcal{B} - \mathcal{L}$ can be of dimension up to $n - 2$, while $\mathcal{B} - \mathcal{L}$ only has the homotopy type of a complex of dimension $\leq n - r - 1$, so that this model can be considered quite wasteful.

To obtain a more efficient model for a matroid M on the ground set $[n]$, let \mathcal{NS} denote the poset of *non-spanning* proper subsets of M ordered by inclusion. In other words, \mathcal{NS} consists of the subsets σ of $[n]$ with matroid rank $\text{rk}(\sigma) < \text{rk}(M)$. Now, as mentioned in Section 1, the matroid closure map

$$\begin{aligned} \kappa : \mathcal{NS} &\rightarrow \mathcal{L}, \\ x &\mapsto \bigvee x \end{aligned}$$

deformation retracts \mathcal{NS} to the geometric lattice \mathcal{L} in \mathcal{B} . We obtain:

Theorem 3.8. *With $(\mathcal{L}, \mathcal{B}, r, n)$ as above, we have*

$$\mathcal{B} \setminus \mathcal{L} \simeq \mathcal{B} - \mathcal{L} \simeq \mathcal{B} - \mathcal{NS}.$$

Moreover, $\mathcal{B} - \mathcal{NS}$ is an efficient model, in the sense that $\dim(\mathcal{B} - \mathcal{NS}) \leq n - r - 1$.

Proof. It remains only to verify the claim on the dimension; this follows immediately once we notice that every element of \mathcal{B} of cardinality $\leq r - 1$ is non-spanning. \square

It follows from the work of Rota that the dimension bound of Theorem 3.8 is tight if \mathcal{L} is not the Boolean lattice.

3.6. Matroid duality is Alexander duality. The dimension of $\mathcal{B} - \mathcal{NS}$ is bounded above by $n - r - 1$, which coincides with the rank of the dual matroid M^* of M . This suggests a connection between $\mathcal{B} - \mathcal{NS} \simeq \mathcal{B} - \mathcal{L}$ and $\mathcal{L}[M^*]$. Indeed, as was pointed out in [Bjö92, Exercise 7.43, p. 278], such a connection is provided by combinatorial Alexander duality [Sta82]. We have,

$$\mathcal{B} - \mathcal{NS} \simeq \{[n] \setminus \sigma : \sigma \text{ spanning in } M\} \cong \{\tau : \tau \text{ independent in } M^*\}$$

The second complex is precisely the combinatorial Alexander dual of \mathcal{NS} , and the last isomorphism follows from standard matroid duality.

It is known that also the poset \mathcal{I} of independent sets of a matroid M is shellable, and in particular $(r - 2)$ -connected, cf. [Bjö92]. Combined with the previous remark, this provides an alternative proof of Corollary 3.7.

Part B. Lefschetz theorems for smooth tropical varieties

4. GEOMETRY AND COMBINATORICS OF POLYHEDRAL COMPLEXES AND SMOOTH TROPICAL VARIETIES.

In this and the following section we review and develop the foundations of the geometry of polyhedral and tropical spaces, including their study via stratified Morse Theory.

4.1. Open polyhedra and topology of restrictions. Let A and B , $B \subset A$, denote two polyhedral complexes such that for every face b of B , there exists a unique minimal face with the property that $a \in A$, $a \notin B$, $b \subset a$. Then \tilde{a} is the *cofacet* of b in A , and $O = A \setminus B$ is an *open polyhedral complex*. This condition implies in particular that a regular neighborhood of B in A is PL homeomorphic to $B \times [0, 1]$, and that A collapses (in the sense of Whitehead) onto $A - B$, cf. [Whi78]. The faces of O are the faces of A , minus the faces of B .

In general, there is little relation between a polyhedral complex and its restrictions. However, the following observation for restrictions of polyhedral complexes is useful to keep in mind for applications of stratified Morse theory.

Proposition 4.1. *Let X denote any compact polyhedral complex in \mathbb{R}^d , and let C denote the complement of some open, convex set K in \mathbb{R}^d . Then $X \cap C = X \setminus K$ deformation retracts onto $X|_C$ provided $X \setminus K$ is compact.*

Proof. If A , B are convex sets (closed and open, respectively) in \mathbb{R}^d with a point of intersection x then $A \setminus B$ deformation retracts onto $\partial A \setminus B$ via restriction of the radial projection

$$\begin{aligned} A \setminus \{x\} &\longrightarrow \partial A \\ y &\longmapsto (x + \text{pos}(y - x)) \cap \partial A. \end{aligned}$$

We can now argue by induction on the dimension of faces of X : We claim that if σ is any facet of X that intersects K , then σ deformation retracts onto $\partial\sigma \setminus K$, and therefore

$$X \cap C = ((X - \sigma) \cap C) \cup (\sigma \cap C)$$

deformation retracts onto $(X - \sigma) \cap C$. With this procedure, we can iteratively remove all faces of X not in C by deformation retractions. The claim follows. \square

4.2. Tangent fan, stars and links. Let $X \subset \mathbb{R}^d$ be any polyhedral space, and let $p \in X$ be any point. Then $T_p X$ is used to denote the tangent fan (or tangent space) of X at p , and $T_p^1 X$ is the restriction of $T_p X$ to unit vectors. If Y is any subspace of X , then $N_{(p,Y)} X$ denotes the subspace of the tangent fan spanned by vectors orthogonal to $T_p Y \subset T_p X$, and we define $N_{(p,Y)}^1 X \stackrel{\text{def}}{=} N_{(p,Y)} X \cap T_p^1 X$.

If X is polyhedral, and σ is any face, then $T_p X$, $T_p^1 X$, $N_{(p,\sigma)} X$ and $N_{(p,\sigma)}^1 X$ are, up to ambient isometry, independent of the choice of p in the relative interior of σ ; we therefore omit p whenever feasible and simply write $T_\sigma X$ etc.

Now, let X be any polyhedral complex, and let σ be any face of X . The *star* of σ in X , denoted by $\text{st}_\sigma X$, is the minimal subcomplex of X that contains all faces of X containing σ . If X is simplicial and v is a vertex of X such that $\text{st}_v X = X$, then X is a *cone* with apex v over the base $X - v$.

Let τ be any face of a polyhedral complex or fan X containing a face σ , and assume that σ is nonempty. Then the set $N_\sigma^1 \tau$ of unit tangent vectors in $N_\sigma^1 X$ pointing towards τ forms a spherical polytope isometrically embedded in $N_\sigma^1 X$.

The collection of all polytopes in $N_\sigma^1 X$ obtained this way forms a polyhedral complex, denoted by $\text{lk}_\sigma X$, the (*combinatorial*) *link* of σ in X . We set $\text{lk}_\emptyset X \stackrel{\text{def}}{=} X$. Motivated by this, we shall also sometimes call $N_\sigma^1 X$ the (*geometric*) *link* of σ in X ; both types of links have the same underlying space, but $\text{lk}_\sigma X$ enjoys additionally a combinatorial structure.

4.3. Morse functions on polyhedral and stratified spaces. If σ is any face of a polyhedral complex, then we call its relative interior σ° a *stratum* (or *cell*). Let now $\tilde{f} : S \rightarrow \mathbb{R}$ denote any function whose domain $S \subset \mathbb{R}^d$ is open, and let X denote any polyhedral space in S . A *critical point* of $f \stackrel{\text{def}}{=} \tilde{f}|_X$ is a critical point of $f|_{\sigma^\circ}$, where $\sigma \in X$ is any face. In other words, x is a critical point of f in X if for the unique stratum σ° of X containing it, we have $\nabla \tilde{f} \perp \text{lin } T_x \sigma^\circ$. *Critical values* are the values of critical points under f . We call f a *Morse function* on X if

- (1) \tilde{f} is smooth on S and every stratum of X .
- (2) $f = \tilde{f}|_X$ is proper, and the critical points of f are finitely many and distinct.
- (3) All critical points are nondegenerate, i.e., for every face $\sigma \in X$, and every critical point $x \in \sigma^\circ$, the Hessian of f_σ at x is non-singular.
- (4) Every critical point is the critical point in a unique stratum, i.e., for every critical point x of f in a stratum σ° , and for every proper coface τ of σ in X , we have $\nabla \tilde{f} \not\perp \text{lin } T_x \tau^\circ$.

For open polyhedral complexes, we simply require that the gradient field is uniformly outwardly oriented at the boundary. Specifically, if $O = A \setminus B$ is an open polyhedral complex, then the restriction $f = \tilde{f}|_O$ of f to O is a *Morse function* on O if

- (1) $\tilde{f}|_A$ is a Morse function on A , and
- (2) for every $b \in B$, and the unique cofacet $a \in A \setminus B$ of b in A ,

$$\langle \nu, \nabla \tilde{f}(x) \rangle < 0$$

for the interior normal ν to b in a .

4.4. The Main Theorem of stratified Morse Theory. With this, we can state the main theorem of stratified Morse theory, specialized to polyhedral spaces (i.e. underlying spaces of polyhedral complexes).

Theorem 4.2 (Goresky–MacPherson [GM88, Part I]). *Let X denote a polyhedral space, and let $f = \tilde{f}|_X : X \rightarrow \mathbb{R}$ be a Morse function on X as above. Then*

- (1) *If $(s, t] \subset \mathbb{R}$ is an interval containing no critical values of f , then $X_{\leq s} \stackrel{\text{def}}{=} f^{-1}(-\infty, s]$ is a deformation retract of $X_{\leq t}$.*
- (2) *Suppose that t is any critical value of f , x the associated critical point and $s < t$ is chosen so that $(s, t]$ contains no further critical values of f . Then, the Morse data at x (and therefore the change in topology from $X_{\leq s}$ to $X_{\leq t}$) is given by the product of tangential and normal Morse data of f at x .*

4.5. Convex superlevel sets. If \tilde{f} has convex superlevel sets, we can easily work out the normal and tangential Morse data.

Lemma 4.3. *With the notation as in Theorem 4.2, let us assume that for every critical value t of f , $\tilde{f}^{-1}[t, \infty)$ is closed and convex. Consider x any critical point of f , let σ° denote its stratum and assume $s < t = f(x)$ is chosen so that $(s, t]$ contains no further critical values of f . Then*

- (1) *the tangential Morse data at x is given by $(\sigma, \partial\sigma)$, and*
- (2) *the normal Morse data at x is given by*

$$(\mathcal{C}(N_\sigma^1(X \cap f^{-1}(-\infty, t])), N_\sigma^1(X \cap f^{-1}(-\infty, t))).$$

Proof. Since $\tilde{f}^{-1}[t, \infty)$ is closed, smooth and convex for every critical value t , the Morse function $f|_{\sigma^\circ}$ takes a minimum at $x \in \sigma^\circ$. Claim (1) follows. Claim (2) holds regardless of the requirement on superlevel sets, cf. [GM88, Part I, Section 3.9]. \square

5. BASIC NOTIONS IN TROPICAL GEOMETRY.

Here we give a brief overview over the essentials of tropical geometry; for more information, we refer to [Gat06, Kat09, MS09, Mik06, RGST05, SS09].

Set $\mathbb{T} \stackrel{\text{def}}{=} [-\infty, \infty) = \mathbb{R} \cup \{-\infty\}$, the tropical numbers. \mathbb{T} is a semiring endowed with the (*tropical*) addition $\oplus : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$ and (*tropical*) multiplication $\odot : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$ defined as

$$a \oplus b \stackrel{\text{def}}{=} \max\{a, b\} \quad \text{and} \quad a \odot b \stackrel{\text{def}}{=} a + b.$$

The *tropical affine space* \mathbb{T}^d of dimension d is the space $[-\infty, \infty)^d$. The *fine sedentarity* $\mathfrak{S} : \mathbb{T}^d \rightarrow 2^{[d]}$ of a point $x \in \mathbb{T}^d$ is the set $\{i \in [d] : x_i = -\infty\}$. The *sedentarity* $\mathfrak{s} : \mathbb{T}^d \rightarrow \mathbb{N}$ is defined as $\mathfrak{s}(x) = |\mathfrak{S}(x)|$. A point resp. set of sedentarity 0 is also called *mobile*. The *mobile part* of a subset A in \mathbb{T}^d is also denoted by $A|_{\mathfrak{m}}$, and we write $A|_{\mathfrak{S}=I}$ to restrict to the subset of fine sedentarity I . In particular, we have a decomposition

$$\mathbb{T}^d = \bigcup_{I \subset [d]} \mathbb{R}^I \times (-\infty)^{[d] \setminus I}.$$

We define *tropical projective d -space* as

$$\mathbb{TP}^d \stackrel{\text{def}}{=} \mathbb{T}^{d+1} \setminus (-\infty)^{d+1} / x \sim \lambda \odot x.$$

Tropical projective space \mathbb{TP}^d can be obtained as a union of $d+1$ copies T_i , $i = 1, \dots, d+1$ of tropical affine space \mathbb{T}^d , restricted to nonpositive coordinates: If $i \in [d+1]$ is any element, then the set $\tilde{S}_i \stackrel{\text{def}}{=} \{x \in \mathbb{T}^{d+1} : x_i = \max_{j \in [d+1]} x_j\}$ projects to the copy T_i of $\mathbb{T}_{\leq 0}^d$ spanned by x_j , $j \in [d+1] \setminus \{i\}$. The notions of sedentarity and mobility therefore naturally extend to tropical projective space.

Remark 5.1. We similarly, but without mention, extend other notions from affine tropical geometry to projective tropical geometry using this decomposition. Therefore, we may restrict the discussion to the affine definitions for the remainder of this introduction to tropical geometry.

5.1. Polyhedral spaces in \mathbb{T}^d . Recall that tropical space \mathbb{T}^d is stratified into copies $\mathbb{R}^I \times (-\infty)^{[d] \setminus I}$, $I \subset [d]$ of euclidean vector spaces. A *d-polyhedron* P in \mathbb{T}^d is the closure of a polyhedron in \mathbb{R}^d , such that for every face $Q = P \cap (\mathbb{T}^J \times (-\infty)^{[d] \setminus J})$, $J \subset [d]$, and every $I \subsetneq J$, we have

$$(1) \quad Q \cap (\mathbb{R}^I \times (-\infty)^{[d] \setminus I}) = \emptyset \quad \text{or} \quad \dim(Q \cap (\mathbb{R}^I \times (-\infty)^{[d] \setminus I})) = \dim Q - |J| + |I|.$$

A *polyhedral complex* Σ in \mathbb{T}^d is a collection of polyhedra in \mathbb{T}^d with the property that the intersection of any two polyhedra is a face of both. The notions of restriction, link, star etc. for polyhedral complexes in \mathbb{R}^d naturally extend to the tropical case. A *hypersurface* in \mathbb{T}^d is a polyhedral complex that partitions its complement into convex pointed subsets, i.e. no component of the complement contains a line.

5.2. Bergman fans of matroids. Associated to every matroid M is the *Bergman fan* [AK06, Ber71, Stu02]. We identify the elements of the ground set $[n]$ of M with a circuit of integer vectors e_1, \dots, e_n in \mathbb{R}^{n-1} , i.e., a family of vectors such that $\sum_{i=1}^n e_i = 0$, each subcollection of the integer lattice spans \mathbb{Z}^d and that have no other linear dependencies. If F is any subset of $[n]$, we define $e_F = \sum_{e_i \in F} e_i$, and if $\mathbf{C} = F < G < H < \dots$ is any chain of flats in \mathcal{L} , then

$$\text{pos}(\mathbf{C}) \stackrel{\text{def}}{=} \text{pos}\{e_F, e_G, e_H, \dots\}.$$

The *Bergman fan* $\mathfrak{F}(M)$ of M is the fan

$$\mathfrak{F}(M) \stackrel{\text{def}}{=} \{\text{pos}(\mathbf{C}) : \mathbf{C} < [n] \text{ increasing chain in } \mathcal{L}\}.$$

We observe a useful property for later:

Lemma 5.2 (Balancing property). *Let \mathbf{C} denote a maximal chain in \mathcal{L} , and let F denote any element of \mathbf{C} . Then \mathfrak{F} is balanced with unit (and in particular positive) weights at $\text{pos}(\tilde{\mathbf{C}})$, $\tilde{\mathbf{C}} \stackrel{\text{def}}{=} \mathbf{C} \setminus F$, i.e.,*

$$\sum_{\substack{G \in \mathcal{L} \\ G \cup \mathbf{C} \text{ maximal chain}}} e_G \in \text{lin}(\tilde{\mathbf{C}}).$$

Proof. This follows at once from the lattice partitioning axiom for the lattice of flats. \square

A second useful observation is to note that every Bergman fan is also locally a Bergman fan with respect to any interior point.

Lemma 5.3. *Let $\mathfrak{F}(M)$ denote a Bergman fan, and let $r = r_F$ denote the ray of a flat F of M . Then $N_r \mathfrak{F}$ and $T_r \mathfrak{F}$ are itself Bergman fans of matroids.*

Note that this does not hold with respect to natural combinatorial structure on the product of Bergman fans; it is a statement about their underlying spaces only. We shall more generally observe:

Lemma 5.4. *The (underlying space of the) product of two Bergman fans $\mathfrak{F}(N)$, $\mathfrak{F}(L)$ is a Bergman fan.*

Proof. The desired matroid is obtained as the parallel connection of N and L , cf. [Bry86]. \square

Proof of Theorem 5.3. Let $M|_F$ denote the restriction of M to F , and let M/F the contraction of F in M . Then $N_r \mathfrak{F}$ is obtained as the product of $\mathfrak{F}(M|_F)$ and $\mathfrak{F}(M/F)$.

As for $T_r \mathfrak{F}$, we remove the flats containing not contained in, or containing F , but add the flat $E \setminus F$. This gives the desired matroid. Alternatively, $T_r \mathfrak{F}$ is obtained as the product of $\mathfrak{F}(\mathcal{B}[2])$ with $N_r \mathfrak{F}$. \square

5.3. Smooth tropical varieties. Smooth tropical varieties were introduced by Mikhalkin [Mik06]. An *integral affine map* $\varphi : \mathbb{T}^n \rightarrow \mathbb{T}^m$ is a map that arises from a well-defined extension of an integral affine map $\tilde{\varphi} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, which in turn is defined as the composition of an integral linear map and an arbitrary translation. An *abstract smooth tropical variety* of dimension n is an abstract polyhedral complex X with charts (U_α, Φ_α) ; $\Phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{T}^{N_\alpha}$ such that

- (1) $\forall \alpha$, V_α is an open subset of $\mathfrak{F}(M) \times \mathbb{T}^{\mathfrak{s}(Y_\alpha)}$, where M is a loopless matroid with $\text{rk}(M) - 1 + \mathfrak{s}(Y_\alpha) = n$, and the map Φ_α is a homeomorphism.
- (2) $\forall \alpha, \alpha'$,

$$\Phi_\alpha \circ \Phi_{\alpha'}^{-1} : \Phi_{\alpha'}(U_\alpha \cap U_{\alpha'}) \longrightarrow \Phi_\alpha(U_\alpha \cap U_{\alpha'}) \subset V_\alpha$$

can be extended to an integral affine map $\varphi : \mathbb{T}^{N_{\alpha'}} \rightarrow \mathbb{T}^{N_\alpha}$.

- (3) the charts are of finite type, i.e., there exists a finite number of open sets (Q_i) such that $\bigcup Q_i = X$, and such that for every Q_i , there is an α such that $Q_i \subset U_\alpha$ and $\Phi_\alpha(Q_i) \subset V_\alpha$.

We refer to the above varieties as “abstract”, and reserve the term *smooth tropical variety* only for those varieties realized as polyhedral complexes in tropical affine or projective space. Of special importance will be the *(tropical) hyperplane*, a hypersurface that arises as the smooth tropical variety of a single, tropical (affine) linear function none of whose coefficients are degenerate. Combinatorially, a tropical hyperplane arises as the Bergman fan of a uniform matroid of rank r on $r + 1$ elements.

Remark 5.5 (Necessity of the integral structure). This is the standard definition of smooth tropical varieties following Mikhalkin, but for our purposes, the integrality assumption in (2) may be dropped. Indeed, the only case of a “tropical Lefschetz Theorem” that depends on the integrality of assumption considered here will turn out to fail regardless of the integral structure (Proposition 11.5).

5.4. Bounded support. To study the geometry of tropical varieties or more generally polyhedral complexes in \mathbb{T}^d using Morse theory, it is sometimes useful to restrict to a “bounded frame” instead of studying the unbounded variety to make clear the smooth structure at infinity.

For a polyhedral complex $X \in \mathbb{T}^d$ we shall therefore sometimes consider the *bounded support* of X . Clearly, there is a positive real number $\delta < \infty$ such that every face of X of sedentarity 0 intersects the box $(-\delta, \delta)^d$. Then we define $X[\delta] \subset \mathbb{T}_\delta^d \stackrel{\text{def}}{=} [-\delta, \delta]^d$ via

$$\sigma \in X \longmapsto \begin{cases} \sigma \cap [-\delta, \delta]^d & \text{if } \mathfrak{S}(\sigma) = \emptyset \\ \sigma \times [\mathbb{T}^{[d] \setminus \mathfrak{S}(\sigma)}] \cap (-\delta)^{\mathfrak{S}(\sigma)} \times [-\delta, \delta]^{[d] \setminus \mathfrak{S}(\sigma)} & \text{else.} \end{cases}$$

In particular, we simply have $|X[\delta]| = |X| \cap [-\delta, \delta]^d$. Up to PL homeomorphism (i.e., up to PL retriangulation and combinatorial isomorphism), the bounded part is a (open) polyhedral complex in \mathbb{T}^d equivalent to X .

6. THE POSITIVE SIDE OF THE BERGMAN FAN.

For the purpose of proving the Lefschetz theorems we need to first recast our Theorem 2.1 in the language of Bergman fans. For the tropical Lefschetz theorems, we are interested in the topology of the restriction $\text{lk}_0 \mathfrak{F}|_{H^+} \simeq T_0^1(\mathfrak{F} \cap H^+)$ of $\text{lk}_0 \mathfrak{F}$ of a Bergman fan \mathfrak{F} to a closed halfspace H^+ , cf. Section 4.1. We have the following corollary of Theorem 2.1.

Lemma 6.1. *Let M denote any finite matroid, let $\mathfrak{F} = \mathfrak{F}(M)$ in $\mathbb{R}^{|M|-1}$ denote its Bergman fan, and let H^+ be a general position closed halfspace with $\mathbf{0} \in \partial H^+$. Then the link $\text{lk}_{\mathbf{0}} \mathfrak{F}|_{H^+}$ is homotopy Cohen–Macaulay of dimension $r - 2$.*

Proof. Let \mathbf{n} denote the interior normal vector to H . Then

$$\omega = (\omega_1, \dots, \omega_n) = (\mathbf{n} \cdot e_1, \dots, \mathbf{n} \cdot e_n)$$

is a generic weight on the elements $[n]$ of M with $\omega \cdot [n] = 0$. With this we have, for every subset σ of $[n]$, that

$$\sigma \in \mathcal{L}^{>0} \iff \omega \cdot \sigma > 0 \iff \mathbf{n} \cdot e_\sigma > 0 \iff e_\sigma \in H^+$$

so that

$$\text{lk}_{\mathbf{0}} \mathfrak{F}|_{H^+} \cong \mathcal{L}^{>0}.$$

The claim hence follows from Theorem 2.1. \square

Similarly we also have, using Theorem 2.1 (or more immediately the work of Björner [Bjö80]), the following topological characterization of the full Bergman fan.

Lemma 6.2. *Let M denote any finite matroid and $\mathfrak{F} = \mathfrak{F}(M)$ in $\mathbb{R}^{|M|-1}$ its Bergman fan. Then the link $\text{lk}_{\mathbf{0}} \mathfrak{F}$ is homotopy Cohen–Macaulay of dimension $r - 2$.*

7. LEFSCHETZ SECTION THEOREMS FOR CELL DECOMPOSITIONS OF TROPICAL VARIETIES.

We are now ready to prove several Lefschetz theorems for tropical varieties. In each section we first recall the classical Lefschetz theorems, and then proceed to prove analogues for tropical varieties.

A crucial ingredient of the Lefschetz Section Theorem of Andreotti–Frankel [AF59] is a vanishing theorem for CW models and Betti numbers of affine varieties.

Theorem 7.1 (Andreotti–Frankel, [AF59]). *Let X denote a smooth affine n -dimensional variety in \mathbb{C}^d . Then X is homotopy equivalent to a CW complex of dimension $\leq n$. In particular, the integral homology groups of X vanish above dimension n .*

The idea for the proof is to use classical Morse theory; the Morse function is given by the distance f from a generic point in \mathbb{C}^d . The theorem then follows from the Main Lemma of Morse theory, together with a simple index estimate for the critical points of f . See Milnor’s book [Mil63] for an excellent exposition.

From this affine theorem, one can deduce the classical Lefschetz Section Theorem. For smooth algebraic varieties and homology groups, this theorem was proven first by Lefschetz [Lef50], and later Andreotti–Frankel [AF59].

Theorem 7.2 (Lefschetz, Andreotti–Frankel, [Lef50, AF59]). *Let X denote any smooth projective algebraic n -dimensional variety in \mathbb{CP}^d , and let H denote a generic hyperplane in \mathbb{CP}^d . Then the inclusion of $X \cap H$ into X induces an isomorphism of integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

This follows directly from the fact that $X \setminus H$ is an affine variety, Theorem 7.1, and Lefschetz duality. Bott, Thom and Milnor then observed that this theorem extends to homotopy groups, and more generally to cell decompositions of the variety.

Theorem 7.3 (Bott, Milnor, Thom, cf. [Bot59, Mil63]). *Let X denote any smooth projective algebraic n -dimensional variety in \mathbb{CP}^d , and let H denote a generic hyperplane in \mathbb{CP}^d . Then X is, up to homotopy equivalence, obtained from $X \cap H$ by successively attaching cells of dimension $\geq n$. In particular, the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of homotopy and integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

The tropical case. Similarly to the Vanishing Theorem for classical projective varieties, the Lefschetz type theorem for affine tropical varieties proved in this section is crucial for deriving Lefschetz theorems for projective varieties. In the tropical realm, the Andreotti–Frankel Vanishing Theorem for affine varieties takes a form similar to the Lefschetz Theorem for projective varieties. The theorem can be stated as follows:

Theorem 7.4. *Let $X \subset \mathbb{T}^d$ be a smooth affine n -dimensional tropical variety, and let H denote a tropical hypersurface to X . Then X is, up to homotopy equivalence, obtained from $X \cap H$ by successively attaching n -dimensional cells.*

By elementary cellular homology and homotopy theory [Hat02, Whi78], we immediately obtain a Lefschetz Section Theorem for homotopy and homology groups:

Corollary 7.5. *Let $X \subset \mathbb{T}^d$ be a smooth, affine n -dimensional tropical variety, and let H denote a tropical hypersurface to X . Then the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.*

Remark 7.6. Throughout, every tropical variety is considered to be endowed with a triangulation. This is of no further use than merely to allow us to analyze varieties using methods from combinatorial and PL topology. In particular, the triangulation at hand often need not be specified. For concreteness, one may for instance consider it endowed with the natural coarsest triangulation obtained from the face structure of the moment polytope, cf. [MS09], or more generally by induction on the dimension, the balancing property and a basic local-to-global convexity theorem [Tie28] that proves the convexity of every coarsest cell.

Since H contains no lines, it parts tropical affine space into pointed polyhedra, it suffices to treat each of the components separately in the proof of Theorem 7.4.

Lemma 7.7. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , and let P denote any closed pointed convex d -polyhedron or d -polytope in \mathbb{T}^d . Then $X \cap P$ is obtained from $X \cap \partial_{\text{m}} P$ by successively attaching cells of dimension n .*

Here $\partial_{\text{m}} P$ is the *mobile part* of ∂P , i.e.,

$$\partial_{\text{m}} P \stackrel{\text{def}}{=} (\partial P)_{\text{m}} = \{x \in \partial P : \mathfrak{s}(x) = 0\}$$

Proof. As usual for the use of Morse theory here, we use the “bounded support” of X for the underlying smooth structure of \mathbb{T}^d , cf. Section 5.4. Consider a continuous function $\tilde{f} : P \rightarrow \mathbb{R}^{\geq 0}$ that

- (1) is smooth in the interior of P ,
- (2) has strictly convex superlevel sets $\tilde{f}^{-1}[t, \infty)$ for every $t > 0$ and
- (3) such that $\tilde{f}^{-1}\{0\} = \partial_{\text{m}} P$.

We may furthermore assume (by perturbing the function) that \tilde{f} is smooth when restricted to any stratum of X , and that the gradient of \tilde{f} is uniformly outwardly oriented at infinity. With this, for every stratum σ° , the critical function $\tilde{f}|_{\sigma^\circ} : \sigma^\circ \rightarrow \mathbb{R}^{\geq 0}$ has at most one critical value (namely, a minimum), cf. Lemma 4.3. Therefore, the critical points of \tilde{f} are distinct, and finite in number, so that \tilde{f} is a proper Morse function on X . We may furthermore assume that the critical values of

$$\tilde{f} \stackrel{\text{def}}{=} \tilde{f}|_X : X \cap P \rightarrow \mathbb{R}^{\geq 0}$$

are distinct. Let $X_{\leq s} \stackrel{\text{def}}{=} X \cap \tilde{f}^{-1}(-\infty, s]$.

Assume now t is a critical value of \tilde{f} , with critical point x . Let $\varepsilon > 0$ be chosen small enough, so that $(t - \varepsilon, t]$ contains only one critical value of \tilde{f} .

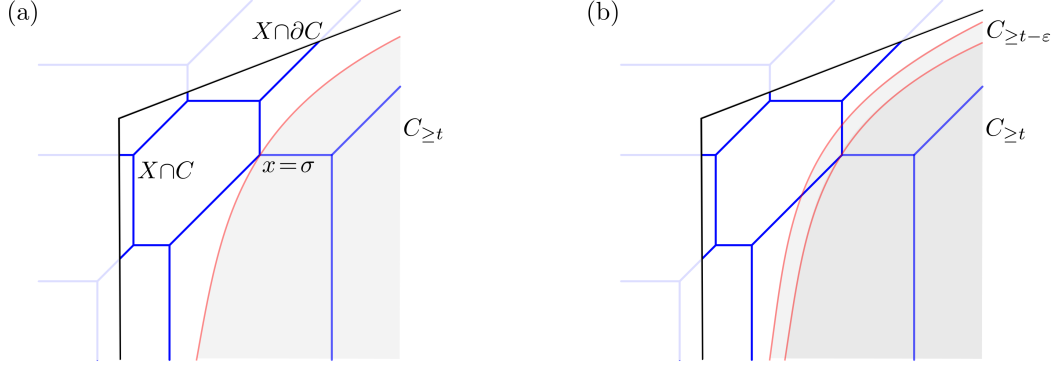


Figure 7.2. Using stratified Morse theory on $X \cap P$, it suffices to consider the Morse data at critical points.

Let σ denote the minimal face of X containing x . The set $P_{\geq t} \stackrel{\text{def}}{=} \tilde{f}^{-1}[t, \infty)$ is a convex set with smooth boundary. By Lemma 4.3(1), the tangential Morse data at x is therefore given by $(\sigma, \partial\sigma)$. Furthermore, considering the halfspace $T_x P_{\leq t}$, the normal Morse data at x is given by $(\mathcal{C} N_{\sigma}^1 X_{\leq t}, N_{\sigma}^1 X_{\leq t})$, where $N_{\sigma}^1 X_{\leq t} = N_{\sigma}^1 X \cap N_{\sigma}^1 f^{-1}(-\infty, t]$ is homotopy equivalent to a wedge of spheres of dimension $(n - \dim \sigma - 1)$ by Lemma 6.1.

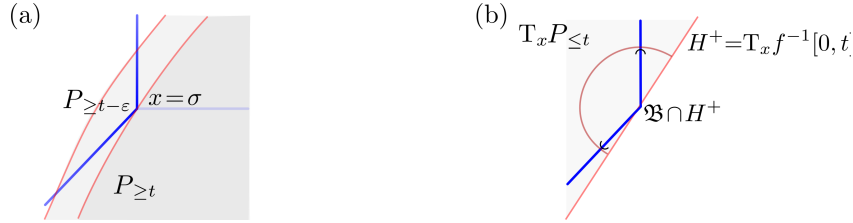


Figure 7.3. The normal Morse data at a critical point $x \in \sigma^\circ$ is given by restricting $\text{lk}_\sigma X \simeq N_{\sigma}^1$ to the hemisphere $T_x^1 P_{\leq t}$.

Therefore, the Morse data at x is given as

$$(2) \quad (\mathcal{C} N_{\sigma}^1 X_{\leq t}, N_{\sigma}^1 X_{\leq t}) \times (\sigma, \partial\sigma) \simeq (\mathcal{C} (N_{\sigma}^1 X_{\leq t} * \partial\sigma), N_{\sigma}^1 X_{\leq t} * \partial\sigma),$$

where $N_{\sigma}^1 X_{\leq t} * \partial\sigma$ is homotopy equivalent to a wedge of $(n - 1)$ -spheres, by Lemma 1.2. The claim now follows with Theorem 4.2(2), finishing the proof of Lemma 7.7. \square

We can now prove the Lefschetz theorem for projective tropical varieties. As in the classical case, it is an easy consequence of the treatment of affine varieties; however, instead of using Lefschetz duality, we can use a direct argument using Morse theory on the projective variety, based on the fact that tropical projective space is a union of tropical affine spaces.

Theorem 7.8. *Let X denote a n -dimensional smooth projective tropical variety in \mathbb{TP}^d , and let $H \in \mathbb{TP}^d$ denote a tropical hypersurface. Then X is, up to homotopy equivalence, obtained from $X \cap H$ by successively attaching cells of dimension n .*

In particular, the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of homotopy and integral homology groups up to dimension $n - 2$, and a surjection in dimension $n - 1$.

Proof. H induces a partition of \mathbb{TP}^d into closed affine pointed polyhedra and polytopes P_i . Now, for every i , we have that $P_i \cap X$ is obtained from $\partial P_i \cap X$ by attaching cells of dimension n by Lemma 7.7. \square

Decomposing the variety, step by step. It is possible to give a more “combinatorial” presentation of the proof of Lemma 7.7 by exhibiting how the cells of a slightly refined version of X are attached, one by one, along the sublevel sets of the Morse function. This is more in line with the Banchoff–Kuehnelt–Kuiper Morse theory for polyhedral complexes (cf. [Ban67, Küh95]) and Forman’s combinatorial Morse Theory, than the stratified Morse theory used above, and it works equally well.

Let σ denote any face of a tropical variety X in \mathbb{T}^d of fine sedentarity S . Let H^+ denote any closed, general position halfspace in $\mathbb{R}^{[d]\setminus S}$ containing σ in its boundary, and let $\tilde{H}^+ \stackrel{\text{def}}{=} H^+ \times \mathbb{T}^S$. Then $\text{st}_\sigma(X \cap \tilde{H}^+)$ and $\text{st}_\sigma X|_{\tilde{H}^+}$ are the *geometric* and *combinatorial half-star* of σ in X with respect to \tilde{H}^+ . With this, we have the following reformulation of Lemma 6.1 in terms of smooth tropical varieties.

Lemma 7.9. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , let σ denote any face of X , and let \tilde{H}^+ be any closed general position halfspace in \mathbb{T}^d with σ in its boundary, as above. Then $\partial \text{st}_\sigma X|_{\tilde{H}^+} = \text{st}_\sigma X|_{\tilde{H}^+} \cap (X|_{\tilde{H}^+} - \sigma)$ is homotopy equivalent to a wedge of spheres of dimension $n - 1$.* \square

For X and P as in Lemma 7.7, let \tilde{X} denote the common refinement of X and P ,

$$\tilde{X} \stackrel{\text{def}}{=} X \cdot P = \{\sigma \cap \tau : \sigma \in X, \tau \in P\}$$

Also, we consider again the Morse function $f : P \rightarrow \mathbb{R}$ defined as the weighted product of the distance functions of the hyperplanes defining facets of a given closed pointed polyhedron P . We then have the following observation:

Proposition 7.10. *Let (\tilde{X}, f) be as above and let $t \geq 0$. Then*

$$\tilde{X} \cap f^{-1}(-\infty, t] \simeq \tilde{X}|_{f_k^{-1}(-\infty, t]}.$$

Proof. Use Proposition 4.1 and the convexity of superlevel sets of \tilde{f} . \square

Corollary 7.11. *If t is a critical value, x the critical point and σ the minimal face of \tilde{X} containing it, and $\varepsilon > 0$ chosen so that $(t - \varepsilon, t]$ contains no critical value besides t , then*

$$\tilde{X}|_{f_k^{-1}(-\infty, t]} - \sigma = \tilde{X}|_{f_k^{-1}(-\infty, t - \varepsilon]}$$

If $0 < t_1 < t_2 < t_3 < \dots$ denotes the sequence of critical values of f , then we call the sequence of complexes

$$X'_j \stackrel{\text{def}}{=} \tilde{X}|_{f_k^{-1}(-\infty, t_j]}$$

a *decomposition sequence* for $\tilde{X}|_P$ with *critical faces* σ_j . Note that then we have $\tilde{X}_{j-1} \simeq \tilde{X}_j - \sigma_j$ and the combinatorial half-star $\text{st}_{\sigma_j} \tilde{X}_j$ is $(n - 1)$ -connected by 7.9. To finalize this section, we obtain a tool to fully replace Lemma 7.7 by a combinatorial decomposition procedure.

Theorem 7.12. *Let X denote a smooth tropical n -dimensional variety in \mathbb{T}^d , and let P denote any closed convex pointed d -polyhedron in \mathbb{T}^d . Then there is a decomposition sequence taking $X \cap P$ to $X \cap \partial_{\text{in}} P$ by iteratively deleting combinatorial half-stars.* \square

8. LEFSCHETZ SECTION THEOREMS FOR COMPLEMENTS OF TROPICAL VARIETIES.

Motivated by the study of complements of subspace arrangements, several Lefschetz theorems were proven that apply to complements of affine varieties, prominently the theorems of Hamm–Lê, cf. [DP03, Ham83, HL71, Lê87].

Theorem 8.1 (Hamm–Lê [Ham83, HL71, Lê87], cf. [DP03, Ran02]). *Let φ denote any non-constant complex polynomial in d variables. If H is a generic hyperplane in \mathbb{C}^d , then $C(\varphi) \stackrel{\text{def}}{=} \{x \in \mathbb{C}^d : \varphi(x) \neq 0\}$ is, up to homotopy equivalence, obtained from $C(\varphi) \cap H$ by attaching cells of dimension d .*

The tropical case. The purpose of this section is to provide a tropical analogue of this influential result. An *almost totally sedentary hyperplane* is the closure of a real hyperplane in \mathbb{T}^d .

Theorem 8.2. *Let X denote a smooth n -dimensional tropical variety in \mathbb{T}^d , and let $C = C(X)$ denote the complement of X in \mathbb{T}^d . Let furthermore H denote an almost totally sedentary hyperplane in \mathbb{T}^d . Then C is, up to homotopy equivalence, obtained from $C \cap H$ by successively attaching $(d - n - 1)$ -dimensional cells.*

In particular, the inclusion of $C \cap H$ into C induces an isomorphism of homotopy groups resp. integral homology groups up to dimension $d - n - 3$, and a surjection in dimension $d - n - 2$.

The central ingredient will be a relative version of Lemma 6.1.

Lemma 8.3. *Let $(M, \mathfrak{F}, H^+, r)$ be as in Lemma 6.1, let H^- denote the closure of $\mathbb{R}^{|M|-1} \setminus H^+$, and let $C = \mathbb{R}^{|M|-1} \setminus \mathfrak{F}$. Then C is, up to homotopy equivalence, obtained from $C \cap H^-$ by attaching cells of dimension $|M| - r - 1$.*

Proof. Let \mathcal{B} denote the Boolean lattice on the ground set of M , and let \mathfrak{F}' denote the associated Bergman fan. Finally, let ω denote the weight associated to H^+ as given in the proof Lemma 6.1, so that

$$\text{lk}_0 \mathfrak{F}|_{H^-} \cong \mathcal{L}^{<0}.$$

Then $\mathbb{R}^{|M|-1} \setminus \mathfrak{F} \simeq \mathfrak{F}' \setminus \mathfrak{F}$ deformation retracts to $\mathcal{B} - \mathcal{L}$, and the retract restricts to a deformation retract of $H^- \setminus \mathfrak{F}$ to

$$\text{lk}_0(\mathfrak{F} \cap H^-) \simeq \mathcal{B}^{<0} - \mathcal{L}^{<0}.$$

Hence, the pair $(C, C \cap H^-)$ is homotopy equivalent to the pair $(\mathcal{B} - \mathcal{L}, \mathcal{B}^{<0} - \mathcal{L}^{<0})$. The claim follows by Corollary 3.6. \square

For the proof, we need stratified Morse theory not for polyhedra, but their complements. The main tools are very similar to those of Section 4.4 and we refer the reader to [GM88, Part I] for details concerning this aspect of stratified Morse theory.

Proof of Theorem 8.2. Without loss of generality (cf. Section 5.4), we may restrict to a bounded box $[-\delta, \delta]^d$ where δ is chosen big enough so that the box $(-\delta, \delta)^d$ intersects all faces of X of sedentarity 0 as well as H . With such a choice of δ , we have

$$C \cap [-\delta, \delta]^d \simeq C \quad \text{and} \quad C \cap H \cap [-\delta, \delta]^d \simeq C \cap H.$$

Let d_H denote the distance from the hyperplane H . Clearly, the function d_H is smooth on every stratum of X . Moreover, if we perturb H by a small amount to a hyperplane H' , then

$$C \cap H' \cap [-\delta, \delta]^d \simeq C \cap H \cap [-\delta, \delta]^d$$

with the additional benefit that $\tilde{f} \stackrel{\text{def}}{=} d_{H'}$ may be assumed to restrict to a Morse function f on X .

We may now apply stratified Morse theory for complements of polyhedra: It suffices to prove that, if x is any critical point of f , and t its value, and $\varepsilon > 0$ chosen small enough such that $[t, t + \varepsilon]$ contains no further critical values of f , then $C_{\leq t+\varepsilon} = C \cap \tilde{f}^{-1}(0, t + \varepsilon]$ is obtained from $C_{\leq t}$ by successively attaching $(d - n - 1)$ -cells.

Now, clearly the minimal stratum of $X[\delta]$ containing x is x itself, so that the tangential Morse data at x is trivial. It remains to estimate the normal Morse data at x . If we set

$$H_x^- \stackrel{\text{def}}{=} \tilde{f}^{-1}[t, \infty) \quad \text{and} \quad H_x \stackrel{\text{def}}{=} \tilde{f}^{-1}\{t\} = \partial H_x^-,$$

this is given by the relative link

$$(\mathbb{T}_x^1(X \cap H_x^-), \mathbb{T}_x^1(X \cap H_x)),$$

i.e., $C_{\leq t+\epsilon}$ is obtained from $C_{\leq t}$ by attaching $\mathbb{T}_x^1(X \cap H_x^-)$ along $\mathbb{T}_x^1(X \cap H_x)$. Since by Lemma 8.3, $\mathbb{T}_x^1 X \cap H_x^-$ is obtained from $\mathbb{T}_x^1(X \cap H_x)$ by successively attaching $(d-n-1)$ -cells, the claim follows by the main theorem for stratified Morse functions. \square

9. TROPICAL HODGE THEORY AND (p, q) -HOMOLOGY.

Tropical (p, q) -homology was introduced by Itenberg–Katzarkov–Mikhalkin–Zharkov [IKMZ]. Despite its name, tropical (p, q) -theory should be thought of as an analogue of Hodge theory in complex algebraic geometry. For more details, we refer the reader to [IKMZ, MZ14, Sha11, Zha13]. Here we discuss tropical Hodge theory only for the case of characteristic 0. For the integral case, see Section 11.4.

9.1. Tropical link and tangent fan. Let X be a polyhedral complex in affine tropical space \mathbb{T}^d , and let σ denote a face of X of fine sedentarity S . Then $\text{tT}_\sigma X \stackrel{\text{def}}{=} \mathbb{T}_\sigma(X \cap (\mathbb{R}^{[d] \setminus S} \times \{-\infty\}^S))$ is the *tropical tangent fan* of σ in X . Let τ be any facet of σ in X in \mathbb{T}^d . Then there is a natural map

$$\text{d}_{\sigma \rightarrow \tau} : \text{tT}_\sigma X \longrightarrow \text{tT}_\tau X.$$

If $\mathfrak{S}(\sigma) = \mathfrak{S}(\tau)$, then $\text{d}_{\tau \rightarrow \sigma}$ is given by natural inclusion of tangent fans. If $\mathfrak{S}(\sigma) \neq \mathfrak{S}(\tau)$, then $\mathfrak{S}(\sigma) \subsetneq \mathfrak{S}(\tau)$ and $\text{d}_{\tau \rightarrow \sigma}$ is given by restriction of the projection along coordinate directions

$$\mathbb{R}^{[d] \setminus \mathfrak{S}(\sigma)} \longrightarrow \mathbb{R}^{[d] \setminus \mathfrak{S}(\tau)}.$$

9.2. p -groups. The coefficients of tropical Hodge theory are given by the p -groups, which form analogues to the sheaf of differential forms in classical Hodge theory.

Definition 9.1 (p -groups). Let Σ denote any polyhedral fan, i.e., any collection of rational polyhedral cones in \mathbb{R}^d pointed at $\mathbf{0}$. For $p \geq 0$, we associate to Σ the subgroup $\mathcal{F}_p \Sigma$ of $\bigwedge^p \mathbb{R}^d$ generated by elements $v_1 \wedge v_2 \wedge \cdots \wedge v_p$, where v_1, v_2, \dots, v_p are real vectors that lie in a common subspace $\text{lin } \sigma, \sigma \in \Sigma$. The groups $\mathcal{F}_p \Sigma$ are called the p -groups.

9.3. Tropical Hodge theory. We give an intuitive definition of tropical cellular Hodge theory, for a more thorough treatment we refer the reader to [IKMZ, MZ14]. Tropical Hodge groups can, alternatively, be defined using generalized singular or simplicial homology theories, but we will stick to a construction based on cellular homology with a local, facewise constant system of coefficients.

Let X denote any tropical variety (realized in tropical affine or projective space, or abstract), or more generally any polyhedral complex in tropical space. If σ is a face of X , and p is a nonnegative integer, then we set $(\mathcal{F}_p X)|_\sigma \stackrel{\text{def}}{=} \mathcal{F}_p(\text{tT}_\sigma X)$. With this, we have the *tropical (p, q) -chains*

$$C_q(X; \mathcal{F}_p) \stackrel{\text{def}}{=} \bigoplus_{\sigma \text{ } q\text{-face of } X} \tilde{H}_q(\sigma, \partial\sigma; \mathcal{F}_p X)$$

where we set $(\mathcal{F}_p X)|_\sigma = \tilde{H}_q(\sigma, \partial\sigma; \mathcal{F}_p X)$. There is a natural boundary map $\partial : C_q(X; \mathcal{F}_p) \rightarrow C_{q-1}(X; \mathcal{F}_p)$ that arises as the composition of the classical cellular boundary map $\tilde{\partial}$, composed with the map $\text{d}_{\sigma \rightarrow \tau}^*$ of p -groups induced by the map $\text{d}_{\sigma \rightarrow \tau}$ defined above, where τ is any facet of σ . This

gives us a chain complex $C_\bullet(\mathcal{F}_p)$; the associated homology groups $H_{(p,q)}(X)$ are the (p, q) -homology groups.

9.4. Some facts in tropical Hodge theory. The (p, q) -homology groups, or tropical Hodge groups, are natural analogues of the classical Hodge groups in algebraic geometry: in [IKMZ] it is proven that for X a smooth projective tropical variety obtained as the limit of a 1-parameter family (X_t) of smooth complex projective varieties, the Hodge groups of a generic fiber X_t are closely related to the tropical Hodge groups of X . Moreover, it follows from classical arguments that, for realizable varieties X , we have the conjugation symmetry $H_q(X; \mathcal{F}_p X) \cong H_p(X; \mathcal{F}_q X)$, cf. [MZ14]. Not everything is analogous to the classical situation though: the (p, q) -homologies do not seem to satisfy the naive analogue of the Hodge Index Theorem [Sha11, Theorem 3.3.5], and the extent to which positivity plays a role in the study of tropical (p, q) -homology is not clear.

Let us close this section by mentioning some useful results to keep in mind.

Theorem 9.2 (cf. [MZ14, Proposition 5], [IKMZ]). *The tropical Hodge groups are independent of the cell-structure of the tropical variety chosen.*

Another instance for such an argument is the following:

Lemma 9.3. *Let X denote any polyhedral fan with cone point v , and let P denote a convex polytope containing v in the interior. Then we have a natural map of chain complexes*

$$C_{q-1}(X \cap \partial P; \mathcal{F}_p X) \longrightarrow C_q(X \cap P, X \cap \partial P; \mathcal{F}_p X)$$

that induces an isomorphism of tropical Hodge groups.

Here, X is considered as a polyhedral complex rather than a fan, so that $\mathcal{F}_p X$ is not constant in general (and in particular not simply the p -group of the fan X).

Proof. The desired map of chains is given by the join operation, which sends a cell $\sigma \in |Y| \cap P$ to the cell $v * \sigma$. It is easy to check that the induced map in homology gives the desired isomorphism. \square

9.5. Transversal chains. It is often useful to assume that every homology class can be assumed to arise from a representative in general position.

Definition 9.4. Let X be a smooth tropical n -variety, and let Σ denote a decomposition of X into polyhedra. We call a (singular) (p, q) -chain c *transversal* if every cell in the k -skeleton of the support of c does not intersect the ℓ -skeleton of Σ , provided $k + \ell < n$. We call a chain in X *transversal* if it is transversal with respect to some triangulation of X .

Lemma 9.5 (cf. [MZ14, Lemma 3]). *Let X denote any n -dimensional smooth tropical variety. Let $c \in C_q(X; \mathcal{F}_p X)$ denote a (p, q) -chain of X such that for some face σ of X , we have $\partial c \cap \text{st}_\sigma^\circ X = \emptyset$. If $q < n$, then there is a (p, q) -chain $\tilde{c} \in C_q(X; \mathcal{F}_p X)$ that*

- (1) *is transversal with respect to X , when restricted to the open star of σ in X ,*
- (2) *is isomorphic to c outside of $\text{st}_\sigma X$, and such that*
- (3) *$(c - \tilde{c})$ is the boundary of a $(p, q + 1)$ -chain of X supported in $\text{st}_\sigma X$.*

Mikhalkin and Zharkov proved that this pushing lemma, that pushes a chain away from a low-dimensional face also holds with respect to integral tropical Hodge theory, in contrast with the directional pushing lemma later.

Let us note two corollaries: The first is that every homology class can be assumed transversal.

Corollary 9.6 ([MZ14, Corollary 2]). *Every cycle in a tropical variety is homologous to a transversal cycle, and every tropical (p, q) -homology class has a transversal representative.*

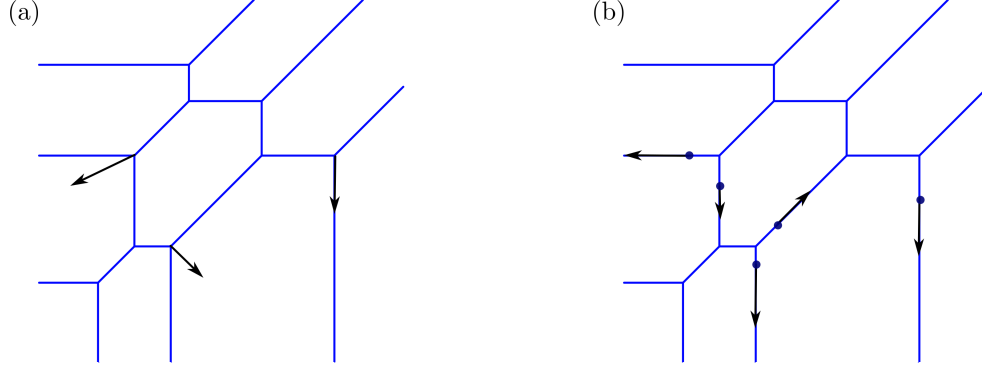


Figure 9.4. Pushing a chain to transversality.

The second consequence is that Lemma 9.5, applied to $(0, q)$ -chains, gives a new proof of Lemma 6.2, at least for the special case of integral homology. We therefore recover the classical result of Folkman [Fol66] on the homology of full geometric lattices.

Corollary 9.7. *The geometric lattice of a rank r matroid is Cohen–Macaulay of dimension $r - 2$.*

10. LEFSCHETZ SECTION AND VANISHING THEOREMS FOR TROPICAL HODGE GROUPS.

The Lefschetz Section Theorem for Hodge groups was first established by Kodaira and Spencer.

Theorem 10.1 (Kodaira–Spencer, c.f. [KS53]). *Let X denote any smooth projective algebraic n -dimensional variety in \mathbb{CP}^d , and let H denote a generic hyperplane in \mathbb{CP}^d . Then the inclusion $X \cap H \hookrightarrow X$ induces a map*

$$H^q(X, \Omega_X^p) \rightarrow H^q(X \cap H, \Omega_{X \cap H}^p)$$

that is an isomorphism provided $p + q \leq n - 2$, and a surjection for $p + q = n - 1$.

For a standard proof of this result, recall that by the Hodge Decomposition Theorem, we have

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Together with the Dolbeault operators, this decomposition is functorial; the result now follows from Theorem 7.2 for complex coefficients. Alternatively, one can prove the theorem directly and algebraically, using the Vanishing Theorem of Akizuki–Kodaira–Nakano:

Theorem 10.2 (Akizuki–Kodaira–Nakano Vanishing Theorem). *Let $X \in \mathbb{CP}^d$ denote a smooth, compact projective n -dimensional variety, and let $L \rightarrow X$ be a positive line bundle. Then*

$$H^q(X, \Omega_L^p) = 0 \text{ for all } p + q > n.$$

Theorem 10.1 then follows from the long exact sequence of Hodge groups and Serre duality, cf. [Voi02].

Remark 10.3. Notice that the logic of the first proof of the Kodaira–Spencer Lefschetz Theorem may also be reversed, and we can conclude Theorem 7.2 for the complex field of coefficients from it. In other words, for smooth complex algebraic varieties, the Lefschetz theorem for Hodge groups is *weaker* than the Lefschetz theorems of Lefschetz, Andreotti–Frankel and Bott–Milnor–Thom.

The tropical case. Contrary to the classical case, the analogous theorems for smooth tropical varieties are not as easily derived. Our Lefschetz Section Theorem for Hodge groups is stated as follows:

Theorem 10.4. *Let X denote any n -dimensional smooth tropical variety in \mathbb{TP}^d , and let $H \subset \mathbb{TP}^d$ denote a hyperplane transversal to X . Then the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of (p, q) -homology up to dimension $p + q \leq n - 2$, and a surjection in dimension $p + q = n - 1$.*

The analogous theorem fails for integral (p, q) -groups, see Section 11.4. For the proof of Theorem 10.4, given at the end of this section, we follow the classical, direct proof of the Kodaira–Spencer Lefschetz Section Theorem. That means, we first prove a tropical analogue of the Akizuki–Kodaira–Nakano Vanishing Theorem; the tropical Lefschetz Section Theorem for Hodge groups 10.4 then swiftly follows.

Theorem 10.5. *Let X denote any n -dimensional smooth tropical variety in \mathbb{TP}^d , and let P denote a pointed polyhedron of codimension k transversal to X . Then*

$$H_q(X \cap P, X \cap \partial_{|\mathfrak{m}} P; \mathcal{F}_p X) = 0 \quad \text{for all } p + q < \dim X \cap P = n - k.$$

Pushing Chains, smooth tropical halflinks and the tropical AKN Theorem 10.5. The idea for the proof is to “push” (p, q) -chains in $X \cap P$ (resp. $X \cap Q$) towards $X \cap \partial_{|\mathfrak{m}} P$ (resp. $X \cap \partial_{|\mathfrak{m}} Q$). This in particular gives us a procedural view on the deformation of chains, and quickly implies the tropical AKN Theorem 10.5 as in Lemma 7.7.

Lemma 10.6 (Pushing chains). *Consider X an n -dimensional smooth tropical variety in \mathbb{TP}^d . Let $c \in C_q(X; \mathcal{F}_p X)$ denote a (p, q) -chain of X such that for some vertex v of X , $c|_{\text{st}_v X}$ is supported in a tropical geometric half-star $\text{st}_v X|_{H^+}$ (where H^+ is a closed halfspace in general position).*

Assume that ∂c is not supported in v . If additionally $p + q < n$, then there is a (p, q) -chain $\tilde{c} \in C_q(X; \mathcal{F}_p X)$ that

- (1) *is supported in $X \setminus \{v\}$,*
- (2) *is isomorphic to c outside of a neighborhood of v in X , and such that*
- (3) *$(c - \tilde{c})$ is the boundary of a $(p, q + 1)$ -chain of X supported in $\text{st}_v X$.*

This lemma generalizes a result of Mikhalkin–Zharkov (Lemma 9.6, compare Section 9.5) which deforms cycles using a similar principle, but without forcing them into a certain direction. As we will see, this last aspect only works in characteristic 0, compare also Proposition 11.5.

The main ingredient of this “Pushing Lemma” will be a version Lemma 6.1, which we reprove for tropical Hodge groups, and the following lemma that establishes that framings (= local coefficient systems) can be pushed past critical strata.

Lemma 10.7. *Consider a Bergman fan \mathfrak{F} of dimension n in \mathbb{R}^d and let H^+ denote a closed general position halfspace whose boundary contains the origin. Then any element of $\mathcal{F}_p \mathfrak{F}$, $p < n$, can be written as a linear sum of elements in $\mathcal{F}_p(\mathfrak{F} \cap H^+)$.*

Proof. The proof of this fact follows a Morse-theoretic argument, induction on p and the balancing property, and we restrict first to the case $p > 1$. We consider the natural stratification of the Bergman fan \mathfrak{F} . Let f denote any generic continuous function $\mathbb{R}^d \rightarrow \mathbb{R}$ whose superlevel sets $f^{-1}[t, \infty)$, $t > 0$ are convex cones pointed at the origin, and that are otherwise smooth. Assume further that f is chosen so that $f^{-1}[0, \infty) = \overline{H \setminus H^+}$.¹

¹Such a function f can be constructed, for instance, as a conjugation of the angular distance from the interior normal vector of ∂H^+ in H^+ with a generic Möbius transformation.

Consider a representative γ of an element in $\mathcal{F}_p \mathfrak{F}$. As we decrease t , we push γ to an equivalent element in $\mathcal{F}_p(\mathfrak{F} \cap H^+)$.

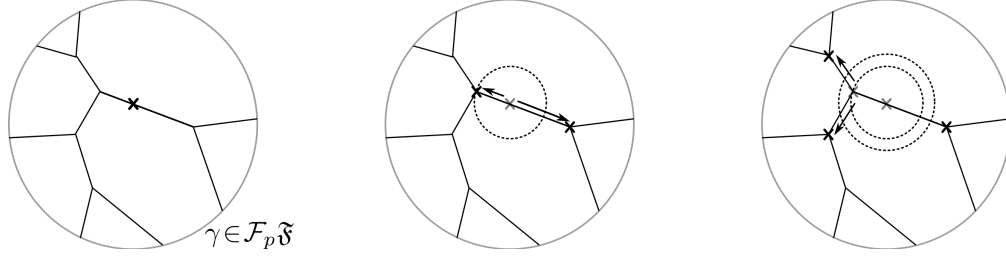


Figure 10.5. We push the generators of a framing γ to an equivalent framing in the positive side of a Bergman fan along a Morse function.

Observe that, for every cone σ of the Bergman fan \mathfrak{F} , the boundary rays $(r_*) \in \mathbb{R}^d$ of σ , span $\text{lin } \sigma$. Hence, we may write γ as a linear combination of exterior products $\bigwedge^p x_{i,j}$, with the property that $x_{1,j} \in r_j$ (where r_j is a ray of \mathfrak{F} , generated by a flat of the underlying matroid). Assume therefore for a moment that γ is brought into this “standard” form.

With this, we conclude that when the level sets of f pass a ray $r = r_F$ corresponding to a flat F of the underlying matroid M , we can assume by induction on p (using the induction assumption since $N_r \mathfrak{F}$ is a Bergman fan) that γ is equivalent to an element in $\mathcal{F}_p(\mathfrak{F} \cap f^{-1}(-\infty, t])$ (where $t = f(r)$). Now, we may reduce the support to an element in $\mathcal{F}_p(\mathfrak{F} \cap f^{-1}(-\infty, t))$ by genericity of f . Hence, the proof of the lemma may be inductively reduced to the case $p = 1$, $n \geq 2$.

To prove the statement for $p = 1$ and any $n \geq 2$, we may assume that $\mathcal{F}_1 \mathfrak{F} \cong \mathbb{R}^d$. It remains to prove that $\mathcal{F}_1(\mathfrak{F} \cap H^+)$ spans \mathbb{R}^d , where H^+ is assumed in generic position, and we may assume that the underlying matroid M is of rank 3 (so that $n = 2$ in the notation above) to prove this fact.

A *subhalfspace* is the intersection of a (linear) hyperplane with a closed halfspace whose boundary is a linear hyperplane transversal to it. A subhalfspace g is *parallel* to a subhalfspace h if $h \cap g = \partial h = \partial g$. Their *angle* $\alpha(g, h) = \alpha(h, g)$ is defined as the oriented dihedral angle, cf. Figure 10.6.



Figure 10.6. To study a geometric halfink, we choose the angle to a subhalfspace h similar to a Morse function.

We need to prove that $\mathfrak{F} \cap H^+$ is not contained in a subhalfspace $h \subset H^+$. Consider α the angular distance of subhalfspaces parallel to h to the halfspace h . Since \mathfrak{F} spans \mathbb{R}^d , there is a maximal angle $\alpha_0 < \pi$ such that,

$$\{g : g \in \alpha^{-1}[0, \alpha_0)\} \cap \mathfrak{F} = h \cap \mathfrak{F}.$$

Then, one of the two subhalfspaces at distance α_0 from h , say g_0 , contains a ray r of \mathfrak{F} in its interior (since H^+ may be assumed in generic position). Moreover, no cone of \mathfrak{F} of dimension ≥ 2 is contained in g_0 by genericity, compare Figure 10.7.

It follows that all ≥ 2 -dimensional cones of \mathfrak{F} incident to r are on one side of the hyperplane spanned by g_0 , and it follows that \mathfrak{F} cannot be balanced with positive weights, contradicting Lemma 5.2. \square

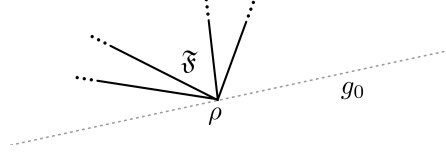


Figure 10.7. Lemma 10.7, $p = 1 < n$ follows from the balancing the property of Bergman fans.

We now prove a (p, q) -analogue for Theorem 2.8.

Lemma 10.8. *Let \mathcal{L} denote the lattice of flats of a matroid on groundset $[m]$ and of rank $r \geq 2$, let \mathfrak{F} denote its Bergman fan, and let ω denote any generic weight on its atoms. Let t denote any real number with $t \leq \min\{0, \omega \cdot [m]\}$. Then, for any compact polyhedron P containing the origin in its interior, we have*

$$H_q(\mathfrak{F}^{>t} \cap P, \mathfrak{F}^{>t} \cap \partial P; \mathcal{F}_p \mathfrak{F}) = 0 \text{ for } p + q \leq r - 2.$$

Here, we use the natural combinatorial isomorphism $\Delta(\mathcal{L}) \xrightarrow{\zeta} \mathfrak{F}/\{\emptyset\}$ that sends a chain \mathbf{C} in \mathcal{L} to the cone $\text{pos}(\mathbf{C})$ to define $\mathfrak{F}^{>t} \stackrel{\text{def}}{=} \zeta(\Delta(\mathcal{L}^{>t}))$.

Moreover, we interpret, in Lemma 10.8 and until the end of its proof, a fan X as a complex, so that $\mathcal{F}_p X$ is generally a non-constant system of coefficients on X rather than simply the p -group of the fan X , cf. Lemma 9.3.

Proof. By Lemma 9.3, it suffices (or more accurately is equivalent) to prove that the groups $H_q(\mathfrak{F}^{>t} \setminus \{\mathbf{0}\}; \mathcal{F}_p \mathfrak{F})$ vanish for $p + q \leq r - 3$.

Consider first the permutahedral fan \mathfrak{P} (i.e. Bergman fans over the Boolean lattice $\mathcal{B} = \mathcal{B}[m]$): By Theorem 2.6, the associated part of the Bergman fan $\mathfrak{P}^{>t}$ is the cone over a $(m - 2)$ -dimensional disk in \mathbb{R}^{m-1} or a complete fan; therefore the (p, q) -groups $H_q(\mathfrak{P}^{>t} \setminus \{\mathbf{0}\}; \mathcal{F}_p \mathfrak{P}^{>t})$ vanish for $p + q \leq n - 3 \geq r - 3$.

Now, we canonically extend ζ to a map $\Delta(\mathcal{B}) \longrightarrow \mathfrak{P}$ (even though we are interested only in the subposet $\mathcal{L} \subset \mathcal{B}$). In abuse of notation, we further extend ζ to a map of posets by defining $\zeta(\mathcal{P}) \stackrel{\text{def}}{=} \zeta(\Delta(\mathcal{P}))$ for a poset $\mathcal{P} \subset \mathcal{B}$.

Since

$$H_q(\mathfrak{P}^{>t} \setminus \{\mathbf{0}\}; \mathcal{F}_p \mathfrak{P}^{>t}) \cong H_q(\mathfrak{P}^{>t} \setminus \{\mathbf{0}\}; \mathcal{F}_p \zeta(\mathcal{B}^{>t} \cup \mathcal{L})),$$

the latter vanish as well under the same conditions.

To prove the general case, we need to observe a Künneth type lemma.

Lemma 10.9. *Let X_1, X_2 denote two fans, and assume that the (p, q) -homology groups of $X_i \setminus \{\mathbf{0}\}$ vanish for $p + q \leq a_i$, $i = 1, 2$. Then the (p, q) -homology of $(X_1 \times X_2) \setminus \{\mathbf{0}\}$ vanishes for $p + q \leq a_1 + a_2 + 2$.*

Proof. The proof reduces to proving that any (p, q) -cycle z in $(X_1 \times X_2) \setminus \{\mathbf{0}\}$ that arises as a join of cycles z_i in $X_i \setminus \{\mathbf{0}\}$, $i = 1, 2$, respectively, is a boundary provided $p + q \leq a_1 + a_2 + 2$. It suffices to prove that one of them, say z_i , is a boundary in $X_i \setminus \{\mathbf{0}\}$, which follows by assumption. \square

Now, we argue as in Theorem 2.8 and Lemma 1.4: To transition from $\mathfrak{P}^{>t}$ (which we understand already) to $\mathfrak{F}^{>t}$, we remove the rays of $\mathfrak{P}^{>t}$ not in $\mathfrak{F}^{>t}$ one by one according to decreasing rank.

Consider an intermediate fan $\mathfrak{F}^{>t} \subset \mathfrak{J} \subset \mathfrak{P}^{>t}$, and r a ray in $\mathfrak{J} - \mathfrak{F}^{>t}$ corresponding to a maximal element $\mu = \zeta^{-1}(r)$ in the intermediate poset $\mathcal{I} = \zeta^{-1}(\mathfrak{J} - \mathfrak{F}^{>t}) \subset \mathcal{B}$. By Lemma 10.9 and induction

on the rank,

$$H_{q-1}(\mathfrak{J} \setminus (\zeta(\mathcal{I} \setminus \{\mu\}) \cup \mathbf{r}); \mathcal{F}_p \zeta(\mathcal{I} \cup \mathcal{L})) = 0$$

for all $p + q \leq r - 2$. Hence, Lemma 10.8 follows with Lemma 9.3 and by a second induction on the number of elements in $\mathcal{B}^{>t} \setminus \mathcal{L}^{>t}$. \square

Proof of Lemma 10.6. First observe that by Lemma 10.7, the framing may be pushed to the halfspace H^+ .

The case of mobile v follows swiftly: By Lemma 10.8, the relative (p, q) -groups $H_q(\text{st}_v X, \partial \text{st}_v X; \mathcal{F}_p X)$ vanish provided $p + q < n$.

Now, for the restriction $c' = c|_{\text{st}_v X}$ of c to $\text{st}_v X$, we have $\text{supp}(\partial c') \subset \partial \text{st}_v X$. Then, by assumption, c' is a boundary in $C_q(\text{st}_v X, \partial \text{st}_v X; \mathcal{F}_p X)$, so that there is a chain $b \in C_{q+1}(\text{st}_v X, \partial \text{st}_v X; \mathcal{F}_p X)$ with $c' + \tilde{c} = \partial b$, where $\text{supp} \tilde{c} \in \text{st}_v X \setminus \{v\}$. Hence, $c - c' + \tilde{c}$ is homologous to c modulo b , and not supported in $\{v\}$.

It remains only to discuss the case where v is of nontrivial sedentarity. By induction on the sedentarity, we may assume that the claim is proven for sedentarity $J \subsetneq I = \mathfrak{S}(v)$.

If we consider any face τ in the support of c' ; then the inclusion

$$\tilde{\tau} \stackrel{\text{def}}{=} \tau \cap (\mathbb{T}^{[d] \setminus I} \times \{-\infty\}^I) \hookrightarrow \tau$$

induces a projection map of local coefficients $(\mathcal{F}_p X)|_\tau \twoheadrightarrow (\mathcal{F}_p X)|_{\tilde{\tau}}$, which we linearly extend to an endomorphism $p_I : (\mathcal{F}_p X)|_{\text{st}_v \tau} \rightarrow (\mathcal{F}_p X)|_{\text{st}_v \tau}$. Using this, we decompose

$$c|_{\text{st}_v X} = \underbrace{c|_{\text{st}_v X} - \ker p_I(c|_{\text{st}_v X})}_{\stackrel{\text{def}}{=} c_{\text{mob}}} + \underbrace{\ker p_I(c|_{\text{st}_v X})}_{\stackrel{\text{def}}{=} c_{\text{sed}}}.$$

Both chains have the property that their boundaries lie outside of $\text{st}_v X$, and hence satisfy the assumptions of the lemma.

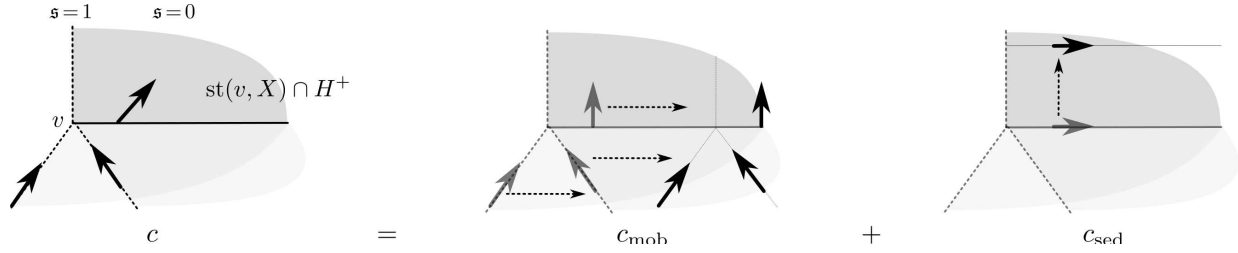


Figure 10.8. At a face of positive sedentarity, we decompose the chain c into two summands c_{sed} and c_{mob} ; the latter is “mobile” in the sense that it can be pushed homologically to faces of strictly smaller sedentarity. The same does not apply to c_{sed} , which can instead be pushed into the interior of H^+ .

Now, c_{sed} may be interpreted as a relative chain (i.e., a chain relative to $\mathbb{T}^{[d] \setminus I} \times \{-\infty\}^I$); it can be deformed to an homologous and otherwise isomorphic chain by pushing it into the intersection of $\text{st}_v X$ and the interior of H^+ in a neighborhood of v , using the same arguments as for the sedentarity 0 case.

The chain c_{mob} , instead, can be deformed by homologically pushing it into a coface of v with strictly smaller sedentarity, and then invoking the induction assumption on the sedentarity. \square

Proof of Theorem 10.5. As in the proof of Lemma 7.7 (whose notation we adopt for simplicity), we may consider a function $\tilde{f} : P \rightarrow \mathbb{R}^{\geq 0}$ with convex superlevel sets that restricts to a Morse

function f on $X \cap P$, so that we are left to analyze the Morse data at critical loci. Geometrically, these are but products of Bergman fans and polyhedra of the appropriate dimension. Specifically, consider a critical value t , some $\varepsilon > 0$ small enough, and the inclusion

$$j_t : X_{\leq t-\varepsilon} \hookrightarrow X_{\leq t},$$

where $X_{\leq s} = X \cap f^{-1}(-\infty, s]$ as usual. If the corresponding critical point lies in the relative interior of a face σ , the Morse data is $(\mathcal{C}(N_\sigma^1 X_{\leq t} * \partial\sigma), N_\sigma^1 X_{\leq t} * \partial\sigma)$, compare identity (2).

By Lemma 10.6 and Lemma 10.9, we may therefore deform any chain (p, q) -chain c in $C_q(X_{\leq t}; \mathcal{F}_p X)$ with $\text{supp } \partial c \in X_{\leq t-\varepsilon}$ and $p + q < n - k$ homologously to a chain \tilde{c} supported in $X_{\leq t-\varepsilon}$ with $\partial \tilde{c} = \partial c$. In particular, every cycle z in $Z_q(X_{\leq t}; \mathcal{F}_p X)$ may be homologously deformed to a cycle \tilde{z} in $Z_q(X_{\leq t-\varepsilon}; \mathcal{F}_p X)$, provided that $p + q < n - k$. Hence, the induced map

$$j_t^* : H_q(X_{\leq t-\varepsilon}; \mathcal{F}_p X) \longrightarrow H_q(X_{\leq t}; \mathcal{F}_p X)$$

is surjective up to dimension $p + q \leq n - k - 1$.

Moreover, if z is a boundary, then \tilde{z} is the boundary of a chain

$$\tilde{c} \in C_{q+1}(X_{\leq t}; \mathcal{F}_p X), \quad \text{supp } \partial \tilde{c} = \text{supp } \tilde{z} \subset X_{\leq t-\varepsilon},$$

and we may homologously deform \tilde{c} to be supported in $X_{\leq t-\varepsilon}$ if $p + q + 1 < n - k$, so that j_t^* is injective up to dimension $p + q \leq n - k - 2$. Iterated application of this argument at the (discrete) set of critical loci gives the desired result. \square

We now turn to the proof of the tropical Kodaira–Spencer Theorem 10.4: We push chains in X to such chains in H , as guaranteed by Theorem 10.5. The resulting cycle is a cycle in $C_q(X \cap H; \mathcal{F}_p X)$. If it also lies in $C_q(X \cap H; \mathcal{F}_p(X \cap H))$ we are done; if not, we iterate the process.

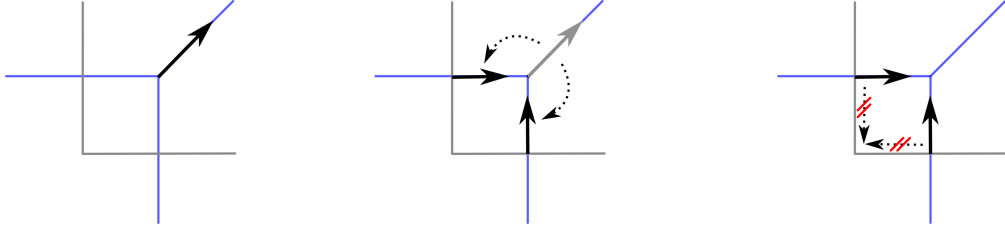


Figure 10.9. A 1-dimensional smooth tropical variety X , part of a tropical hypersurface H variety and a $(1, 0)$ chain c in it; while c can be pushed to $X \cap H$, it can not be pushed so that it is supported with coefficients in $\mathcal{F}_p(X \cap H)$.

Note that while the first pushing step might succeed, it might not work in general to pass from $C_q(X \cap H; \mathcal{F}_p X)$ to $C_q(X \cap H; \mathcal{F}_p(X \cap H))$, cf. Figure 10.9.

Proof of the tropical Kodaira–Spencer Theorems 10.4. We assume that $p \geq 1$, noting that the case when $p = 0$ was dealt with already (since $\mathcal{F}_0 \equiv \mathbb{Z}$). Moreover, we work with the natural coarsest cell structure on the hypersurface H . We divide the proof into two parts by showing that the maps

$$(I) \quad H_q(X \cap H; \mathcal{F}_p X) \longrightarrow H_q(X; \mathcal{F}_p X)$$

and

$$(II) \quad H_q(X \cap H; \mathcal{F}_p(X \cap H)) \longrightarrow H_q(X \cap H; \mathcal{F}_p X)$$

induced by inclusion are isomorphisms for $p + q < n - 1$, and onto for $p + q \leq n - 1$. For this, we show that

(I) every relative cycle $c \in Z_q(X, X \cap H; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(X \cap H; \mathcal{F}_p X)$ and

(II) every chain $c \in C_q(X \cap H; \mathcal{F}_p X)$ is homologous to a chain $\tilde{c} \in C_q(X \cap H; \mathcal{F}_p(X \cap H))$

as long as $p + q < n$.

Now, Claim (I) is immediate from Theorem 10.5, since H divides \mathbb{T}^d resp. \mathbb{TP}^d into polyhedra to which we can apply Theorem 10.5 separately.

For Claim (II), let us notice first that by Lemma 9.6, we can assume every chain to be transversal to the skeleton of X , compare Lemma 9.5. We may write c as

$$c = \sum_{\sigma: q\text{-cell} \rightarrow X} v^\sigma, \quad v^\sigma \in (\mathcal{F}_p X)_{|\sigma}.$$

There are now three situations to consider:

- (1) If $c \in C_q(X \cap H; \mathcal{F}_p(X \cap H))$, there is nothing to prove.
- (2) Let σ denote any facet in the support of c , and let h denote the minimal face of H that contains σ . We may assume, by transversality, cf. Lemma 9.5, that the relative interior of σ lies in the relative interior of the intersection of h with a facet S of X . Then $H' \stackrel{\text{def}}{=} (H \cap \text{aff } S) / \text{aff } h$ is a tropical hyperplane of dimension $\text{codim } h - 1$ and with conept $\sigma' \stackrel{\text{def}}{=} \sigma / \text{aff } h$.

Lemma 10.10. *Consider \mathfrak{F} in \mathbb{R}^{m-1} the Bergman fan of a uniform matroid of rank r on groundset $[m]$. Then $\mathcal{F}_p \mathfrak{F} \cong \mathcal{F}_p \mathbb{R}^{m-1}$ for all $p \leq r - 1$.*

Proof. This follows at once from [Zha13, Theorem 4], or alternatively directly from induction on the rank and deletion/contraction, which manifest as tropical modifications on Bergman fans, cf. [Sha13]. To start the induction, note that the case $m = r$ is trivial. \square

If h is of codimension at least p in H , then, by above lemma,

$$(\mathcal{F}_p H')_{|\sigma'} \cong (\mathcal{F}_p \text{aff } H')_{|\sigma'} \cong (\mathcal{F}_p \text{aff } S / \text{aff } h)_{|\sigma'}$$

and therefore $(\mathcal{F}_p(H \cap S))_{|\sigma} = (\mathcal{F}_p X)_{|\sigma}$. The restriction $c|_\sigma$ therefore lies in $C_q(X \cap H; \mathcal{F}_p(X \cap H))$.

- (3) With σ and h as in the previous step, let us assume that h is of codimension $\ell - 1 < p$ in H . Then we can write

$$c|_h = \sum_{k \geq 0}^\ell \sum_{\sigma: q\text{-cell} \rightarrow X} w_{p-k}^\sigma \wedge y_k^\sigma,$$

where $w_\alpha^\sigma \in (\mathcal{F}_\alpha(X \cap H))_{|\sigma}$ and $y_\beta^\sigma \in (\mathcal{F}_\beta X)_{|\sigma} / (\mathcal{F}_\beta(X \cap H))_{|\sigma}$. Analogously to the argument for the previous case, we see that only the case of summand $k = \ell$ is nontrivial. Then

$$\tilde{c}_{|h,\ell} \stackrel{\text{def}}{=} \sum_{\sigma: q\text{-cell} \rightarrow X} w_{p-\ell}^\sigma \in C_q(X \cap H; \mathcal{F}_{p-\ell}(X \cap H)).$$

Now, $\tilde{c}_{|h,\ell}$ is homologous to a chain $\tilde{c}'_{|h,\ell}$ in $C_q(\partial h; \mathcal{F}_{p-\ell}(X \cap H))$ as long as

$$(p - \ell) + q < \dim X \cap h = \dim X - \ell = n - \ell,$$

by Theorem 10.5. Since H is transversal to X , we hence conclude that there exists a chain $c'_{|h,\ell}$ in $C_q(\partial h; \mathcal{F}_p X)$ which is homologous to

$$c_{|h,\ell} \stackrel{\text{def}}{=} \sum_{\sigma: q\text{-cell} \rightarrow X} (w_{p-\ell}^\sigma \wedge y_\ell^\sigma) \cdot \sigma \in C_q(X \cap H; \mathcal{F}_{p-\ell}(X \cap H)).$$

Iterating this argument, we see that we can find a chain homologous to c in a face of codimension at least p , and the desired conclusion follows from the previous step. \square

11. TROPICAL LEFSCHETZ THEOREMS: REMARKS, EXAMPLES AND OPEN PROBLEMS.

11.1. Theorems of Lefschetz type in tropical geometry. Classical Lefschetz Theorems are often phrased abstractly, using the notions of ample divisors and positive line bundles, cf. [Laz04, Voi02]. In tropical geometry there seem presently to be no such notions that are generally agreed upon, cf. [Car13]. Another instance of a Lefschetz Section Theorem is the Artin–Grothendieck Vanishing Theorem [Laz04, Theorem 3.1.13] for constructible sheaves. Again, no analogous notion is known to exist for tropical varieties.

A worthwhile long-range goal for this line of research could be to understand the Hard Lefschetz Theorem in the tropical setting, cf. [Del80].

There is a plethora of questions: *Define tropical line bundles and divisors. Do these notions give rise to Lefschetz Theorems for abstract smooth tropical varieties? What is the connection to intersection rings of wonderful models? What is the tropical analogue of the Vanishing Theorem of Artin–Grothendieck? What is a tropical Kähler manifold? Does it satisfy a tropical analogue of the Hard Lefschetz Theorem? What about the Hodge–Riemann relations?*

11.2. Lefschetz-type theorems for general tropical varieties and balanced complexes. It is not hard to see that the Lefschetz-type section theorems do not apply to general tropical varieties and balanced complexes (cf. [MS09]). In fact, the Lefschetz property breaks down already for simple, balanced 2-dimensional fans in \mathbb{R}^4 , as shown by the union X of any two transversal 2-planes in \mathbb{R}^4 . While X is contractible, $X \cap H$ is disconnected for every hyperplane $H \in \mathbb{R}^4$. We conclude that the map

$$\mathbb{Z}^2 \cong H_0(X \cap H; \mathbb{Z}) \longrightarrow H_0(X; \mathbb{Z}) \cong \mathbb{Z}$$

is not the required isomorphism.

So, the intuition that the Lefschetz-type theorem might hold for general balanced complexes is wrong. However, it is not totally unjustified: the Lefschetz Theorem holds, almost trivially, for all balanced hypersurfaces and their complements.

Proposition 11.1. *Let X and H denote two balanced hypersurfaces in \mathbb{R}^d . Then X is, up to homotopy equivalence, obtained from $X \cap H$ by successively attaching $(d - 1)$ -dimensional cells.*

The reasoning requires a little more technology than what is useful here; we therefore content ourselves with giving a sketch of the argument:

Proof. By basic cellular topology, we see that X must be Cohen–Macaulay. Moreover, we see that if C is any component of the complement of H in \mathbb{R}^d , then $C \cdot \Sigma_X$ is a regular subdivision of C , and therefore shellable relative to the boundary ∂C by the Bruggesser–Mani rocket shelling [BM71] and Alexander duality of shellings [AB12]. Hence, X also shells to $X \cap H$, and the relative space $(X, X \cap H)$ is Cohen–Macaulay as well. \square

11.3. A stronger version of the Pushing Lemma? It is natural to wonder whether it is sufficient in Lemma 10.6 to assume only that $q < n$ (as opposed to the more restrictive $p + q < n$). This would reconcile 10.6 with the Mikhalkin–Zharkov Transversality Lemma 9.5. However, this is not the case, as is shown by the following counterexample.

Consider the uniform matroid U_4^3 of rank 3 on 4 elements, let (e_i) denote an integral circuit in \mathbb{R}^3 and let \mathfrak{F} denote the Bergman fan of U_4^3 spanned by (e_i) . Furthermore, let H^+ be a general position closed halfspace.

Then there exists a $(1, 1)$ -chain c such that ∂c does not intersect some open neighborhood O of x , and which admits no $(1, 1)$ -chain \tilde{c} that (1) is supported in $\mathfrak{F} \setminus O$, (2) is isomorphic to c outside of O

and (3) $c - \tilde{c}$ is the boundary of a $(1, 2)$ -chain supported in \overline{O} . Equivalently, there is a $(1, 0)$ -cycle z which is supported in $T_0^1(\mathfrak{F} \cap H^+)$, and which is not the boundary of a $(1, 1)$ -chain in the same domain.

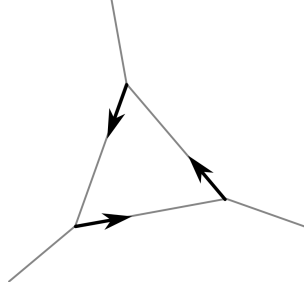


Figure 11.10. A representative for a nontrivial homology $(1, 0)$ -class in the tropical halflink $T_0^1(\mathfrak{F} \cap H^+)$ of the Bergman fan associated to U_4^3 .

Such a chain is easy to construct: without loss of generality we assume that H^+ contains e_1, e_2 and e_3 in the interior, and consider $\sigma_i \stackrel{\text{def}}{=} \text{pos } e_i, i \in \{1, 2, 3\}$ (we may disregard the unboundedness of the cells since we are only interested in a small neighborhood of the origin). Let $\{1, 2, 3\}$ be ordered cyclically, and consider the $(1, 1)$ -chain

$$c \stackrel{\text{def}}{=} \sum_{i=1}^3 (e_{i-1} - e_i) \sigma_i.$$

Clearly, if we consider the faces $\tau_{i,j} \stackrel{\text{def}}{=} \text{pos}\{e_i, e_j\}, i \neq j \in \{1, 2, 3\}$, then a chain

$$\gamma \stackrel{\text{def}}{=} \sum_{i=1}^3 (a_i \cdot (e_i - e_{i+1}) + b_i \cdot e_{i+1}) \tau_{i,i+1},$$

with $c - \partial\gamma \notin O$, where O is any bounded open neighborhood of the origin, must satisfy

$$b_i + a_{i+1} + b_{i+1} = 0 \quad \text{and} \quad a_{i+1} - b_{i+1} = 0 \quad \text{and} \quad a_i = 1 \quad \text{for all } i \in \{1, 2, 3\}.$$

This is inconsistent, so that the cycle c generates a nontrivial homology class. To illustrate this counterexample, see Figure 11.10 for the corresponding picture in the halflink.

11.4. An integral tropical Kodaira–Spencer Theorem. Tropical Hodge theory was originally defined by Mikhalkin [IKMZ] over the integers. The impact of this notion on the Lefschetz Theorem is briefly discussed here.

Definition 11.2 (Integral p -groups and integral tropical Hodge groups). Let Σ denote any polyhedral fan. For $p \geq 0$, we associate to Σ the subgroup $\mathbf{F}_p \Sigma$ of $\bigwedge^p \mathbb{Z}^d$ generated by elements $v_1 \wedge v_2 \wedge \cdots \wedge v_p$, where v_1, v_2, \dots, v_p are integer vectors that lie in a common subspace $\text{lin } \sigma, \sigma \in \Sigma$. The groups $\mathbf{F}_p \Sigma$ are also known as the *integral p -groups* of Σ .

In analogy with the case of characteristic 0 Hodge groups, the integral tropical Hodge (p, q) -groups are naturally the homology groups generated by the local system of coefficients $\mathbf{F}_p \Sigma$.

Example 11.3. For every matroid of rank at least 2 (on groundset $[n]$), we have $\mathbf{F}_1 \mathfrak{F} \cong \mathbb{Z}^{n-1}$.

Example 11.4 (Orlik–Solomon algebras and integral p -groups). For a Bergman fan $\mathfrak{F} = \mathfrak{F}(M)$, the corresponding co-groups $\mathbf{F}^p \mathfrak{F}$ are, quite naturally, a graded algebra $\mathbf{F}^\bullet(\cdot)$. For every matroid M , we then have a natural isomorphism between $\mathcal{OS}^\bullet(M)$, the projectivized Orlik–Solomon algebra of Σ , and $\mathbf{F}^\bullet(\mathfrak{F}(M))$, the graded algebra of co- p -groups of the Bergman fan of M [Zha13].

It is natural to ask whether the Kodaira–Spencer Lefschetz Section Theorem holds for integral (p, q) -groups. Consider this problem:

Let X denote any n -dimensional smooth tropical variety in \mathbb{TP}^d , and let $H \subset \mathbb{TP}^d$ denote a hyperplane transversal to X . Is it true that the inclusion $X \cap H \hookrightarrow X$ induces an isomorphism of (p, q) -homology up to dimension $p + q \leq n - 2$, and a surjection in dimension $p + q = n - 1$?

The answer is clearly negative, essentially because $X \cap H$ may be badly non-smooth. However, even if one gives strong assumptions on the intersection of X and H , the problem fails to admit a positive answer, because the integral version of the pushing lemma is wrong:

Proposition 11.5 (Pushing chains fails for integral Hodge theory). *There exists a Bergman fan \mathfrak{F} of dimension 2 (pointed at $\mathbf{0}$) and a closed general position halfspace H^+ in general position w.r.t. \mathfrak{F} such that $\mathbf{F}_1(\mathfrak{F} \cap H^+)$ is a strict subgroup of $\mathbf{F}_1\mathfrak{F}$.*

Proof. Let M denote the Fano plane on groundset $\{1, 2, 3, 4, 5, 6, 7\}$ and with rank two flats

$$\{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$

Then $\mathbf{F}_1\mathfrak{F}(M) = \mathbb{Z}^6$. Let $\omega = (4, 4, 4, -3, -3, -3, -3)$, and let H^+ be the halfspace determined by the interior normal ω . Then

$$\mathcal{L}^+(M) = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}.$$

To compute $\mathbf{F}_1(\mathfrak{F} \cap H^+)$ we now have to compute the integral span of

$$\{\sigma \in \mathfrak{F} : \sigma \text{ incident to } \text{Im } \mathcal{L}^+(M) \subset \mathfrak{F}\}.$$

Now, every cone is spanned by the rays of the generating flats, so that $\mathbf{F}_1(\mathfrak{F} \cap H^+)$ is generated by the flats

$$G \stackrel{\text{def}}{=} \mathcal{L}^+(M) \cup \{\{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$

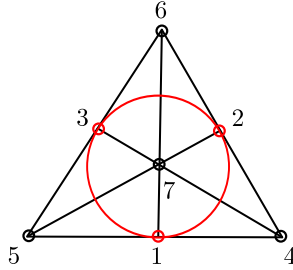


Figure 11.11. The tropical pushing lemma for tropical Hodge groups exhibits torsion on the Fano Plane. The positive part of the weighted geometric lattice is red.

Consider now the weight vector $\vartheta = (0, 0, 0, 1, 1, 1, 1)$. Then

$$\vartheta \cdot S = 0 \pmod{2}$$

for every $S \in G$. But, say, $\vartheta \cdot \{4\} = 1$, so $\mathbf{F}_1(\mathfrak{F} \cap H^+)$ is a strict subgroup (of index 2) in $\mathbf{F}_1\mathfrak{F}$. \square

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